On boundedness and completion in Nonstandard Analysis

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October 24, 2016

Abstract

In this paper, we investigate the topological and the algebraic structures of convex subrings of ${}^*\mathbb{R}$, a nonstandard extension of $\mathbb{R}$. Next, using convex rings, we define several kinds of complex bounded polynomials and under some additional assumptions, we prove that the quasi-standard part of such polynomial provides an entire function over some nonarchimedean field extension of $\mathbb{C}$. Finally, we define new bounded points in a nonstandard extension of a topological vector space, and we construct new nonstandard hulls of topological vector spaces and we show that such spaces are complete. Some examples are provided to illustrate our construction.

KEYWORDS: Nonstandard analysis, Bornology, Nonstandard hull, internal polynomials
AMS CLASSIFICATION: 54J05, 26E35, 03H05, 46S20, 46S10, 12J25

1 Introduction

The methods of nonstandard analysis have been applied to topology with illuminating and satisfying results, see Robinson [18] and Luxemburg[16, 15]. They provide an alternative to the classical description of a topological space by open sets. The notion of *monad* is a fundamental concept which encodes a topology and most of the subsequent development comes from their properties. Meanwhile, constructing nonstandard hulls turned out to be an effective method for

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obtaining new mathematical objects from those available. For metric spaces, this was carried out by Robinson. For normed spaces and uniform spaces, this was accomplished by Luxemburg and by Henson and Moore[7] for topological vector spaces. In the case of measure spaces, it is the Loeb spaces that play the role of nonstandard hulls [8]. Roughly speaking, the nonstandard hull is the quotient of the set of "bounded" elements by the equivalence relation of being infinitely close. The latter property is related to a topology whereas being bounded is associated to a *bornology*. Bornologies axiomatize an abstract notion of bounded sets and are introduced as collections of subsets satisfying a number of properties. A bornological space is a set equipped with a bornology. Bornological spaces form a category, the morphisms of which are those functions which preserve bounded sets. The theory of bornological spaces plays an important role in functional analysis, see H. Hogbe-Nlend[9, 10] and Bourbaki [1].

In this paper, we construct bornologies on ${}^*X$, a nonstandard extension of a bornological space $(X, \mathcal{B})$. Likewise the topological framework, where at least two topologies were established by Robinson [18], we will construct two bornologies on ${}^*X$, the first is called the Q-bornology and generated by ${}^*\mathcal{B}$ and the second the S-bornology which is generated by $({}^*B)_{B\in\mathcal{B}}$. For instance in ${}^*\mathbb{R}$, balls with positive real radii generate the S-bornology of ${}^*\mathbb{R}$ and balls with arbitrary positive radii generate the Q-bornology of ${}^*\mathbb{R}$ and both of these bornologies are compatible with the ring structure of ${}^*\mathbb{R}$.

Using nonstandard analysis, we make more concrete the analogy between the topological and the bornological context. In the bornological framework, ${}^bX := \bigcup_{B\in\mathcal{B}} {}^*B$, the set of *bounded points* in ${}^*X$, will encode the bornology $\mathcal{B}$. More precisely, a set is bounded if and only if its nonstandard extension is included in the set of bounded points and a function is bounded if and only if its nonstandard extension preserves bounded points. Thus the set of bounded points in the bornological setting is the counterpart of the monad in the topological framework.

Furthermore, new bornologies are constructed for metric spaces and locally convex spaces using convex subrings of ${}^*\mathbb{R}$. Recently, the first author [11] established new topologies parametrized by convex subrings of ${}^*\mathbb{R}$, called QS-topologies, between the S-topology and the Q-topology.

The paper is organized as follows, section 2 provides a background necessary for the comprehension of the paper. In section 3, we study convex subrings of ${}^*\mathbb{R}$ and we show that they are simply extensions of ${}^b\mathbb{R}$, the ring of bounded numbers in ${}^*\mathbb{R}$. This study involves a comparison between the order topology and the valuation topology on $\widehat{\mathbb{F}} = \mathbb{F}/{}^i\mathbb{F}$, the nonstandard hull of $\mathbb{F}$, where $\mathbb{F}$ is a convex subring of ${}^*\mathbb{R}$, and ${}^i\mathbb{F}$ denotes its maximal ideal. We also provide a complete description of the maximal and the prime spectrum of ${}^b\mathbb{R}$, see Corollary 3.23. In section 4, using convex subrings of ${}^*\mathbb{C}$, we define several kinds of bounded polynomials, and we show that under the assumption, that $\widehat{\mathbb{F}}$ has a nontrivial real-valued valuation, the quasi-standard part of an $\mathbb{F}$-bounded polynomial gives an entire analytic function over $\widehat{\mathbb{F}}^n$. More generally, we prove that the quasi-standard part of an $\mathbb{F}$-bounded polynomial is a generalized power series indexed by a monoid containing $\mathbb{N}$, see Theorem 4.16. These theorems extend some of our previous results obtained in [12], in which it is shown that the standard part of a bounded polynomial, i.e., sending ${}^b\mathbb{C}^n$ to ${}^b\mathbb{C}$, gives an entire function over $\mathbb{C}^n$, where ${}^b\mathbb{C}$ stands for the ring of bounded elements in ${}^*\mathbb{C}$. The proofs are not a direct transposition of those in [12] and substantial modifications are needed. The main reason is that a hyperfinite product of elements in $\mathbb{F}$ of length in $\mathbb{F}$ is not in general in $\mathbb{F}$. This fact constitute one of the fundamental disparity between ${}^b\mathbb{C}$ and $\mathbb{F}$, a proper subring of ${}^*\mathbb{C}$, such that $\mathbb{F} \supsetneq {}^b\mathbb{C}$. In section 5, we construct new topologies on ${}^*E$, a nonstandard extension of a topological vector space E. Then, we define the set of $\mathbb{F}$-bounded points and we construct $\widehat{E}$, the $\mathbb{F}$-nonstandard hull of E. The space $\widehat{E}$ endowed with the quotient topology has the structure of a topological vector space over $\widehat{\mathbb{F}}$. We note that if $\mathbb{F} = {}^b\mathbb{R}$ or ${}^b\mathbb{C}$, then $\widehat{E}$ is the classical nonstandard hull of E constructed by Henson and Moore in [7]. Finally, we provide some examples of $\mathbb{F}$-nonstandard hulls of locally convex spaces.

2 Preliminaries

This section of preliminary notions provides a background necessary for the comprehension of the paper.

2.1 Nonstandard Analysis.

The approach to nonstandard analysis that we use in the present paper follows that of Stroyan and Luxemburg[19]. One starts with a superstructure $V(S) = \bigcup V_n(S)$ over set S, which is often not specified explicitly but chosen large enough to contain all objects under the consideration, real numbers, necessary vector spaces, etc. We suppose that for the enlargement ${}^*S$ of the set S of basic elements, the natural embedding ${}^* : V(S) \to V({}^*S)$ satisfies the following principles:

The extension principle. ${}^*s = s$ for all $s \in S$.

The transfer principle. Let $\Phi(x_1, x_2, \ldots, x_n)$ be a bounded formula of the superstructure $V(S)$ and let $A_1, A_2, \ldots, A_n$ be elements of the superstructure $V(S)$.
Then the assertion $\Phi(A_1, A_2, \ldots, A_n)$ about elements of $V(S)$ holds true iff the assertion $\Phi({}^*A_1, {}^*A_2, \ldots, {}^*A_n)$ about elements of ${}^*(V(S))$ does.

Let ${}^*(V(S))$ be a nonstandard enlargement of a superstructure $V(S)$. An element $x \in {}^*(V(S))$ is called *standard* if $x = {}^*X$ for some $X \in V(S)$; *internal* if $x \in {}^*X$ for some $X \in V(S)$; *external* if x is not internal.

It is well known that a nonstandard enlargement ${}^*(V(S))$ of $V(S)$ can be chosen so that the following principle is satisfied, see for instance Goldblatt [6].

The general saturation principle. If a family $\{A_\gamma\}_\gamma \in \Gamma$ of internal sets possesses the finite intersection property and card(Γ) < card(V(S)).
Then $\bigcap_{\gamma\in\Gamma} A_\gamma \neq \phi$.

In the sequel, we always deal with nonstandard enlargements satisfying the general saturation principle (they are also called *polysaturated*).

2.2 Bornological spaces

A bornological space is a type of space which, in some sense, possesses the minimum structure needed to address questions of boundedness of sets and functions, in the same way that a topological space possesses the minimum structure needed to address questions of continuity.

Definition 2.1. *Let X be a set. A bornology on X is a collection $\mathcal{B}$ of subsets of X such that*

(i) *B is stable under inclusions, i.e., if $A \in \mathcal{B}$ and $A' \subset A$, then $A' \in \mathcal{B}$;*

(ii) *B is stable under finite unions, i.e., if $B_1, \dots, B_n \in \mathcal{B}$, then $\bigcup_{i=1}^n B_i \in \mathcal{B}$.*

We say that the bornology $\mathcal{B}$ is covering if $X = \bigcup_{B\in\mathcal{B}} B$.

Elements of the collection $\mathcal{B}$ are called bounded sets. The pair $(X, \mathcal{B})$ is called a bornological set. A *base of the bornology* $\mathcal{B}$ is a subset $\mathcal{B}_0$ of $\mathcal{B}$ such that each element of $\mathcal{B}$ is a subset of an element of $\mathcal{B}_0$.

If X, Y are bornological sets, a function $f : X \to Y$ is said to be *bounded* if $f(B)$ is bounded in Y for every bounded B in X. One obtains a category of bornological sets and bounded maps.

We say that a bornology $\mathcal{B}_1$ is coarser than the bornology $\mathcal{B}_2$ on X if $\mathcal{B}_1 \subset \mathcal{B}_2$, that is, the identity $(X, \mathcal{B}_1) \to (X, \mathcal{B}_2)$ is bounded.

2.2.1 Examples

(i) If X is a T_1 topological space, there is a bornology consisting of all precompact subsets of X (subsets whose closure is compact). Any continuous map is bounded with respect to this choice of bornology.

(ii) If X is a metric space, there is a bornology where a set is bounded if it is contained in some open ball. Any Lipschitz map is bounded with respect to this choice of bornology.

(iii) If X is a measure space, then the subsets of the sets of finite measure form a bornology.

2.3 Valuation rings

In this section, we present some basic definitions of the valuation theory of commutative fields.

Recall that a subring R of a field K is called a valuation ring if every element of K is either in R or is the inverse of an element of R.

Proposition 2.2. *Let R be a valuation ring in its field of fractions K. Then*

(i) *M is an R-submodule of K if and only if $RM \subset M$,*

(ii) *the set of R-submodules of K is totally ordered by the inclusion.*

Using inversion in the field K, we can define a bijection from the chain of R-submodules of K to itself. Indeed, let M be a non-empty set of K. We define

$$^{i}M = \{x \in K : x = 0 \text{ or } 1/x \notin M\}.$$

Obviously, M is an R-submodule of K if and only if ^{i}M is an R-submodule of K and $^{ii}M = M$ holds for any R-submodule M of K.

Theorem 2.3. *[2] The mapping $^{i} : M \mapsto {}^{i}M$ yields an anti-correspondence between the followings chains*

(i) *R-submodules of K,*

(ii) *R-submodules of K containing R and the chain of proper ideals of R,*

(iii) Spec(R)*, the set of all prime ideals of R, and the set of all subrings of K containing R.*

Indeed, if S is a subring of K containing R, then ^{i}S is the maximal ideal of S, thus it is a prime ideal of R.

3 Convex subrings of $^*\mathbb{R}$

Let $^*\mathbb{R}$ be a nonstandard extension of the field of real numbers $\mathbb{R}$ and $^{i}\mathbb{R}$, $^{b}\mathbb{R}$ and $^{\infty}\mathbb{R}$ stand for the sets of infinitesimals, bounded (or finite) numbers and infinitely large numbers in $^*\mathbb{R}$, respectively. For a comprehensive introduction to nonstandard analysis, the reader is referred to [6, 8, 13, 19].

First, we recall the definition and some properties of convex subrings of $^*\mathbb{R}$.

Definition 3.1. *Let $\mathbb{F}$ be a nonempty subset of $^*\mathbb{R}$. We say that $\mathbb{F}$ is a convex in $^*\mathbb{R}$ if*

$$(\forall x \in {}^*\mathbb{R})(\forall \xi \in \mathbb{F})(|x| \leq |\xi| \Rightarrow x \in \mathbb{F}).$$

Remark 3.2. *There is a one-to-one correspondence between convex subrings of $^*\mathbb{C}$ and those of $^*\mathbb{R}$: let $\mathbb{F}'$ be a convex subring of $^*\mathbb{C}$, then $\mathbb{F} = \mathbb{F}' \cap {}^*\mathbb{R}$ is a convex subring of $^*\mathbb{R}$. Conversely, let $\mathbb{F}$ be a convex subring of $^*\mathbb{R}$, then $\mathbb{F}' = \{a \in {}^*\mathbb{C} : |a| \in \mathbb{F}\}$ is a convex subring of $^*\mathbb{C}$.*

Using the fact that any subring of ${}^*\mathbb{R}$ contains $\mathbb{Z}$, it is clear that if $\mathbb{F}$ is a convex subring of ${}^*\mathbb{R}$, then $\mathbb{F}$ contains ${}^b\mathbb{R}$. We prove that the converse remains true.

Proposition 3.3. *If M is a ${}^b\mathbb{R}$-submodule of ${}^*\mathbb{R}$, then M is convex in ${}^*\mathbb{R}$.*

Proof. Let $x \in {}^*\mathbb{R}$ and $\xi \in M \setminus \{0\}$ such that $|x| \leq |\xi|$. Thus $x/\xi \in {}^b\mathbb{R}$, and we deduce that $x = (x/\xi).\xi \in {}^b\mathbb{R}.M \subset M$, that is, M is convex in ${}^*\mathbb{R}$. $\square$

Corollary 3.4. *Let $\mathbb{F}$ be a subring of ${}^*\mathbb{R}$. Then $\mathbb{F}$ is convex if and only if $\mathbb{F}$ contains ${}^b\mathbb{R}$.*

Therefore, any convex subring $\mathbb{F}$ of ${}^*\mathbb{R}$ is a valuation ring. For the remainder of this paper we fix the following notations : ${}^i\mathbb{F}$ denotes the maximal ideal of $\mathbb{F}$, and ${}^a\mathbb{F} = \mathbb{F} \setminus {}^i\mathbb{F}$, the set of appreciable elements of $\mathbb{F}$ and ${}^\infty\mathbb{F} = {}^*\mathbb{R} \setminus \mathbb{F}$.

Let us recall some properties of ${}^i\mathbb{F}$.

Proposition 3.5. *Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{R}$. Then*

(i) *${}^i\mathbb{F}$ consists of infinitesimals only i.e., ${}^i\mathbb{F} \subset {}^i\mathbb{R}$.*

(ii) *${}^i\mathbb{F}$ is a convex ideal in $\mathbb{F}$ i.e., if $x \in \mathbb{F}$ and $\xi \in {}^i\mathbb{F}$, $(|x| \leq |\xi| \Rightarrow x \in {}^i\mathbb{F})$.*

(iii) *$\mathbb{F}$ is a field if and only if $\mathbb{F} = {}^*\mathbb{R}$.*

We give some examples of convex subrings of ${}^*\mathbb{R}$.

3.0.1 Examples

(i) (Finite Numbers). The ring of bounded nonstandard real numbers ${}^b\mathbb{R}$ is a convex subring of ${}^*\mathbb{R}$. Its maximal ideal is ${}^i\mathbb{R}$, the set of infinitesimals.

(ii) (Nonstandard Real Numbers). The field of the real numbers ${}^*\mathbb{R}$ is (trivially) a convex subring of ${}^*\mathbb{R}$. Its maximal ideal is $\{0\}$.

(iii) (Robinson Rings). Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$. The ring of the ρ-moderate nonstandard numbers is defined by

$$M_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \rho^{-n} \text{ for some } n \in \mathbb{N}\}.$$

M_ρ is a convex subring of ${}^*\mathbb{R}$. For its maximal ideal we have

$$N_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \rho^{n} \text{ for all } n \in \mathbb{N}\}.$$

(iv) (Logarithmic-Exponential Rings) Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let E_ρ be the smallest convex subring of ${}^*\mathbb{R}$ containing all iterated exponentials of ρ^{-1}, that is,

$$E_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \exp_n(\rho^{-1}) \text{ for some } n \in \mathbb{N}\},$$

where $\exp_0(x) = x$ and $\exp_n(x) = \exp(\exp_{n-1}(x))$ for $x \in {}^*\mathbb{R}$ and $n > 0$. The maximal ideal of E_ρ is

$$ {}^iE_\rho = \{x \in {}^*\mathbb{R} : |x| \le \frac{1}{\exp_n(\rho^{-1})} \text{ for all } n \in \mathbb{N}\}. $$

(v) Let ω be an infinite positive number in ${}^*\mathbb{R}$.

$$ P_\omega = \{x \in {}^*\mathbb{R} : |x| \le n^\omega \text{ for some } n \in \mathbb{N}\}, $$

P_ω is a convex subring of ${}^*\mathbb{R}$, its maximal ideal is given by

$$ {}^iP_\omega = \{x \in {}^*\mathbb{R} : |x| \le \frac{1}{n^\omega} \text{ for all } n \in \mathbb{N}\}. $$

One can easily check that $P_\omega = M_{\exp(-\omega)}$.

The previous three examples can be generalized using *filtrations* of $\mathbb{F}$ by ${}^b\mathbb{R}$-submodules of ${}^*\mathbb{R}$. Our treatment is similar but different from the classical theory of filtrations.

Let $(F_n)_{n\in\mathbb{Z}}$ be a *decreasing* sequence of ${}^b\mathbb{R}$-submodules of ${}^*\mathbb{R}$ such that

$$ \ldots \subset F_{n+1} \subset F_n \ldots \subset F_1 \subset F_0 = {}^b\mathbb{R} \subset F_{-1} \subset \ldots F_{-n} \subset F_{-n-1} \subset \ldots \subset {}^*\mathbb{R} $$

In particular, $(F_n)_{n\ge 1}$ is a sequence of proper ideals of ${}^b\mathbb{R}$, so $F_n \subset {}^i\mathbb{R}$, for all $n \ge 1$.

Assume that the sequence $(F_n)_{n\in\mathbb{Z}}$ satisfy the following two conditions:

(F1) For all $n, m \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $F_n\, F_m \subset F_k$.

(F2) For $n > 0$, $F_{n+1} \subset {}^iF_{-n} \subset F_n$.

One can easily check that the condition (F1) is equivalent to for every $n > 0$, there exists $k > 0$ such that $F_{-n}^2 \subset F_{-k}$.

Let

$$ \mathbb{F} = \bigcup_{n\in\mathbb{Z}} F_n. $$

Obviously, $\mathbb{F}$ is a ${}^b\mathbb{R}$-submodule of ${}^*\mathbb{R}$ containing ${}^b\mathbb{R}$. The condition (F1) yields $\mathbb{F}$ is a subring of ${}^*\mathbb{R}$ extension of ${}^b\mathbb{R}$, hence $\mathbb{F}$ is a valuation ring and its maximal ideal is given by

$$ {}^i\mathbb{F} = {}^i\left(\bigcup_{n\in\mathbb{N}} F_{-n}\right) = \bigcap_{n\in\mathbb{N}} {}^iF_{-n} = \bigcap_{n\in\mathbb{N}} F_n $$

and the last equality follows from the condition (F2).

Typical examples of filtrations satisfying the conditions (F1) and (F2) are given by principal fractional ideals generated by asymptotic scales. The reader is referred to Astrada and Kanwal[4] for the classical definition of asymptotic sequence of functions, and to Jones [5] and to Van den Berg [21] for the non-standard treatment of asymptotics.

Definition 3.6. *[11] A sequence $(\lambda_n)_{n\in\mathbb{N}}$ of infinitesimal positive numbers (except possibly $n = 0$) is called an asymptotic scale if it satisfies the following conditions:*

(i) *for all $n \in \mathbb{N}$, $\frac{\lambda_{n+1}}{\lambda_n} \in {}^i\mathbb{R}$,*

(ii) *for every $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $\lambda_k \leq \lambda_n^2$.*

The sequence $(\lambda_n)_{n\in\mathbb{N}}$ extends to $(\lambda_n)_{n\in\mathbb{Z}}$ by putting

$$\lambda_{-n} = \frac{1}{\lambda_n} \quad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

Let F_n be the principal fractional ideal generated by λ_n, that is, for $n \in \mathbb{Z} \setminus \{0\}$

$$F_n := {\lambda_n}^b\mathbb{R}.$$

One can easily check that if (λ_n) is an asymptotic scale, then $(F_n)_{n\in\mathbb{Z}}$ is a chain of decreasing ${}^b\mathbb{R}$-submodules of ${}^*\mathbb{R}$ satisfying the conditions (F1) and (F2). Indeed, $F_{-n} = F_n^{-1}$ and ${}^iF_{-n} = {\lambda_n}^i\mathbb{R}$. Hence

$$\mathbb{F} = \bigcup_{n\in\mathbb{Z}} F_n = \{x \in {}^*\mathbb{R} : x \in {\lambda_{-n}}^b\mathbb{R} \text{ for some } n \in \mathbb{N}\}$$

is a convex subring of ${}^*\mathbb{R}$ and its maximal ideal is given by

$${}^i\mathbb{F} = \bigcap_{n\in\mathbb{N}} F_n = \{x \in {}^*\mathbb{R} : x \in {\lambda_n}^b\mathbb{R} \text{ for all } n \in \mathbb{N}\}.$$

Using convex subrings of ${}^*\mathbb{R}$, a variety of fields $\widehat{\mathbb{F}}$ is constructed by Todorov[20]. These fields are called $\mathbb{F}$-asymptotic hulls and their elements $\mathbb{F}$-asymptotic numbers. This construction can be viewed as a generalization of A. Robinson's theory of asymptotic numbers, see Lightstone-Robinson [14].

Definition 3.7. *Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{R}$. The $\mathbb{F}$-asymptotic hull is the factor ring $\widehat{\mathbb{F}} = \mathbb{F}/{}^i\mathbb{F}$.*

Let $\widehat{\mathrm{st}} : \mathbb{F} \longrightarrow \widehat{\mathbb{F}}$ stand for the corresponding quotient mapping, called the *quasi-standard mapping.*

If $x \in \mathbb{F}$, we shall often write $\widehat{x}$ instead of $\widehat{\mathrm{st}}(x)$ for the quasi-standard part of x.

We can define an order relation in $\widehat{\mathbb{F}}$, inherited from the order in ${}^*\mathbb{R}$, by

$$\widehat{x} \leq \widehat{y} \text{ if there are representatives } x, y \text{ with } x \leq y.$$

Using the convexity of $\mathbb{F}$, the following proposition is straightforward

Proposition 3.8. $(\widehat{\mathbb{F}}, \leq)$ *is a completely ordered field.*

We note that Todorov[20] proved a strong form of Proposition 3.8 claiming that $\widehat{\mathbb{F}}$ is a real closed field, that is, $\widehat{\mathbb{F}}$ is elementarily equivalent to the real numbers. In other words, it has the same first-order properties as the reals: any sentence in the first-order language of fields is true in $\widehat{\mathbb{F}}$ if and only if it is true in the reals.

3.0.2 Topologies on $\mathbb{F}$-asymptotic hulls

Let $\mathbb{F}$ be a ring extension of ${}^b\mathbb{R}$ filtered by $(F_n)_{n\in\mathbb{Z}}$, a decreasing sequence of ${}^b\mathbb{R}$-modules, satisfying the conditions (F1) and (F2).

For $x \in \mathbb{F}$, let v be the *order function* defined by

$$v(x) = \sup\{n \in \mathbb{Z} : x \in F_n\} \in \mathbb{Z} \cup \{+\infty\}.$$

If v is known then so are the F_n, for F_n is the set of x such that $v(x) \geq n$. The following equivalences directly follow from the definition of v:

(i) $v(x) = p$ if and only if $x \in F_p$ and $x \notin F_{p+1}$,

(ii) $v(x) = +\infty$ if and only if $x \in {}^i\mathbb{F} = \bigcap_{n\in\mathbb{N}} F_n$.

Besides, for $x, y \in \mathbb{F}$, we have

(iii) $v(x - y) \geq \min(v(x), v(y))$,

(vi) $v(\lambda x) \geq v(x)$, for every $\lambda \in {}^b\mathbb{R}$,

(v) if $x - y \in {}^i\mathbb{F}$, then $v(x) = v(y)$.

Using the property (v), we define $\widehat{v}$, the order function on $\widehat{\mathbb{F}}$ by

$$\widehat{v}(a) := v(x)$$

where x is any representative of $a \in \widehat{\mathbb{F}}$, and the previous properties hold for all elements in $\widehat{\mathbb{F}}$.

Remark 3.9. *If we replace the condition* (F1) *by the usual condition of the filtration, that is, for all* $n, m \in \mathbb{Z}$, $F_n\, F_m \subset F_{m+n}$, *we obtain* $v(xy) \geq v(x) + v(y)$, *for all* $x, y \in \mathbb{F}$. *Hence* $\widehat{v}$ *is a pseudo-valuation on* $\widehat{\mathbb{F}}$.

Let now

$$\delta(a,b) := \widehat{\mathbb{F}} \times \widehat{\mathbb{F}} \to [0,+\infty) \ : \ (a,b) \mapsto \mathrm{e}^{-\widehat{v}(a-b)}.$$

The previous properties of v show that $(\widehat{\mathbb{F}}, \delta)$ is an *ultrametric* space. Moreover, $\widehat{F_n} := F_n/{}^i\mathbb{F}$ is a fundamental system of neighborhoods of zero for this metric topology. There is also another natural topology on $\widehat{\mathbb{F}}$ induced by the order.

Theorem 3.10.

(i) *The ultrametric topology is coarser than the order topology on* $\widehat{\mathbb{F}}$.

(ii) *If* F_n *is generated by an asymptotic scale* (λ_n), *then these two topologies coincide. Moreover, the intervals* $(-\widehat{\lambda_n}, \widehat{\lambda_n})$ *is a fundamental system of neighborhoods of zero.*

(iii) *The metric space* $(\widehat{\mathbb{F}}, \delta)$ *is complete.*

Proof. (i) Let $n \in \mathbb{N}$ and r be a positive element of $F_n \setminus F_{n+1}$. We have $(-r,r) = r(-1,1) \subset F_n\, {}^b\mathbb{R} \subset F_n$. Hence $(-\widehat{r}, \widehat{r}) \subset \widehat{F_n}$.

(ii) For any $r \in {}^a\mathbb{F}_+$, there exists $n \in \mathbb{N}$ such that $r \notin F_n$, hence $r > \lambda_n$, that is, $(-\lambda_n, \lambda_n) \subset (-r, r)$. This shows that the order topology coincides with the ultrametric topology. Moreover, the intervals $(-\widehat{\lambda_n}, \widehat{\lambda_n})$ is a fundamental system of neighborhoods of zero, since for any $n \in \mathbb{N}$, $(-\lambda_{n+1}, \lambda_{n+1}) \subset F_{n+1} \subset (-\lambda_n, \lambda_n)$.

(iii) The completeness of $\widehat{\mathbb{F}}$ is proved in [11]. □

Definition 3.11. *We say that a valuation* v *is compatible with the order of* $\widehat{\mathbb{F}}$ *if it satisfies*

$$\forall\, x, y \in \widehat{\mathbb{F}} : \ |x| \le |y| \implies v(x) \ge v(y).$$

Proposition 3.12. *A valuation* v *is compatible with the order of* $\widehat{\mathbb{F}}$ *if and only if* ${}^b\mathbb{R} \subset \{x \in \mathbb{F} : v(\widehat{x}) \ge 0\}$.

Proof. $\implies$: Let $x \in {}^b\mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $|x| \le n$, hence $|\widehat{x}| \le n$. Since the valuation v is compatible with the order of $\widehat{\mathbb{F}}$, we obtain $v(\widehat{x}) \ge v(n) \ge 0$.

$\Longleftarrow$: Let $x, y \in \widehat{\mathbb{F}}$ such that $0 < |x| \le |y|$, then there exists x_1 and y_1 in ${}^a\mathbb{F}$, representatives of x and y respectively such that $|x_1| \le |y_1|$. This shows that x_1/y_1 is bounded. Thus $v(x/y) \ge 0$, that is, $v(x) \ge v(y)$. □

From the inclusion ${}^b\mathbb{R} \subset \{x \in \mathbb{F} : v(\widehat{x}) \ge 0\}$, we obtain $\{x \in \mathbb{F} : v(\widehat{x}) > 0\} \subset {}^i\mathbb{R}$. Hence ${}^a\mathbb{R} \subset \{x \in \mathbb{F} : v(\widehat{x}) = 0\}$, where ${}^a\mathbb{R} = {}^b\mathbb{R} \setminus {}^i\mathbb{R}$.

Proposition 3.13. *If* $\widehat{\mathbb{F}}$ *has a nontrivial real-valued valuation compatible with the order of* $\widehat{\mathbb{F}}$, *then the order topology coincides with the valuation topology on* $\widehat{\mathbb{F}}$.

Proof. Let v be a nontrivial real-valued valuation on $\widehat{\mathbb{F}}$. Then there exists $a \in \widehat{\mathbb{F}}_+$, such that $0 < v(a) < +\infty$. Hence $\{x : |x| \leq a^n\} \subset \{x : v(x) \geq nv(a)\}$, this shows that the valuation topology is coarser than the order topology. Now, let $\alpha \in \widehat{\mathbb{F}}_+$ and $r := v(\alpha)$. Then $\{x : v(x) > r\} \subset \{x : |x| < \alpha\}$. □

Proposition 3.14. *If $\widehat{\mathbb{F}}$ has a nontrivial real-valued valuation v, then*

(i) $\{x \in \mathbb{F} : v(\widehat{x}) \geq 0\}$ *is a maximal subring of* $\mathbb{F}$.

(ii) *Moreover, if v is compatible with the order, then*

$$\mathbb{F} = \bigcup_{n\in\mathbb{N}} \frac{1}{\alpha^n}\, {}^b\mathbb{R} \text{ and } {}^i\mathbb{F} = \bigcap_{n\in\mathbb{N}} (-\alpha^n, \alpha^n)$$

where α is any element in ${}^a\mathbb{F}_+$, such that $0 < v(\widehat{\alpha}) < +\infty$.

Proof. (i) Let $\mathbb{G}$ be a subring of $\mathbb{F}$ such that $\mathbb{G} \supsetneq \{x \in \mathbb{F} : v(\widehat{x}) \geq 0\}$. Then there exists $b \in \mathbb{G}$ such that $v(\widehat{b}) < 0$. Since ${}^i\mathbb{F} = \{x \in \mathbb{F} : v(\widehat{x}) = +\infty\} \subset \mathbb{G}$, we have only to show that ${}^a\mathbb{F} \subset \mathbb{G}$. Let $x \in {}^a\mathbb{F}$, then $v(\frac{\widehat{x}}{\widehat{b}^n}) = v(\widehat{x}) - nv(\widehat{b}) > 0$ for n large. Hence $\frac{x}{b^n} \in \mathbb{G}$, that is, $x \in b^n\mathbb{G}$. Thus $\mathbb{F} \subset \mathbb{G}$.

(ii) Let $\alpha \in {}^a\mathbb{F}_+$ such that $0 < v(\widehat{\alpha}) < +\infty$. Hence $\alpha \in {}^i\mathbb{R}_+ \setminus {}^i\mathbb{F}$. Let $x \in {}^a\mathbb{F}$, then there exists $n \in \mathbb{N}$ such that $v(1/\widehat{\alpha}^n) < v(\widehat{x})$. By compatibility of the valuation v with the order, we get $|\widehat{x}| < \frac{1}{\widehat{\alpha}^n}$. Thus $|x| < \frac{1}{\alpha^n}$. □

Example 3.15. *The Robinson valuation $v(x) = \mathrm{st}(\log_\rho(|\mathrm{x}|))$ is compatible with the order of ${}^\rho\mathbb{R}$.*

3.1 Naturals in $\mathbb{F}$

For studying internal polynomials sending $\mathbb{F}$ to $\mathbb{F}$, see section 4.1, we expect that the following two natural properties are satisfied: hyperfinite sums and products of elements in $\mathbb{F}$ of length $N \in \mathbb{F}_+ \cap {}^*\mathbb{N}$ belong to $\mathbb{F}$. These two properties are satisfied when $\mathbb{F} = {}^b\mathbb{R}$, since ${}^b\mathbb{R} \cap {}^*\mathbb{N} = \mathbb{N}$. Whereas in the general case, hyperfinite products of elements in $\mathbb{F}$ are not in $\mathbb{F}$. The main reason for that the exponential function is not $\mathbb{F}$ stable, that is, there exists $a \in \mathbb{F}$ and $\exp(a) \notin \mathbb{F}$.

The aim of this section is to define the set of naturals in $\mathbb{F}$, denoted by ${}^\mathbb{F}\mathbb{N}$, where the mentioned properties of the sums and the products are satisfied. Next, we introduce $\mathbb{F}^\#$, a convex subring of $\mathbb{F}$, such that $\exp(\mathbb{F}^\#) \subset \mathbb{F}$ and $\mathbb{F}^\# \cap {}^*\mathbb{N} = {}^\mathbb{F}\mathbb{N}$.

Definition 3.16. *We define the set of naturals in $\mathbb{F}$ by*

$${}^\mathbb{F}\mathbb{N} = \{\nu \in {}^*\mathbb{N} : \forall R \in \mathbb{F}_+,\ R^\nu \in \mathbb{F}\}$$

where $\mathbb{F}_+ = \mathbb{F} \cap {}^*\mathbb{R}_+$, the set of non-negative elements of $\mathbb{F}$.

The following properties are straightforward

Proposition 3.17.

(i) ${}^{\mathbb{F}}\mathbb{N}$ *is a semiring.*

(ii) *Let* (x_i) *be an internal sequence in* $\mathbb{F}$ *and* $N \in {}^{\mathbb{F}}\mathbb{N}$. *Then the product* $\prod_{i=1}^{N} x_i$ *is in* $\mathbb{F}$.

(iii) $\mathbb{N} \subset {}^{\mathbb{F}}\mathbb{N} \subset \mathbb{F} \cap {}^*\mathbb{N}$.

Proposition 3.18.

(i) *If* $\widehat{\mathbb{F}}$ *has a nontrivial real-valued valuation compatible with the order of* $\widehat{\mathbb{F}}$, *then* ${}^{\mathbb{F}}\mathbb{N} = \mathbb{N}$.

(ii) *If* $\widehat{\mathbb{F}}$ *is an exponential field (i.e.,* $e^x \in \mathbb{F}$ *for any* $x \in \mathbb{F}$*), then* ${}^{\mathbb{F}}\mathbb{N} = \mathbb{F} \cap {}^*\mathbb{N}$.

Proof. (i) Let $R \in \mathbb{F}_+$ such that $v(\widehat{R}) < 0$, hence $R > 1$. Let $\nu \in {}^{\mathbb{F}}\mathbb{N}$ and assume that $\nu \in {}^{\infty}\mathbb{N}$. Thus $R^\nu \geq R^n$ for any $n \in \mathbb{N}$. Since the valuation is compatible with the order, we obtain $v(\widehat{R^\nu}) \leq nv(\widehat{R})$ for any $n \in \mathbb{N}$, a contradiction. It follows that $\nu \in \mathbb{N}$.

(ii) Let $\nu \in \mathbb{F} \cap {}^*\mathbb{N}$, and $R \in \mathbb{F}_+$, we have $0 \leq R^\nu \leq e^{R\nu}$. By convexity, we get $R^\nu \in \mathbb{F}$, i.e., $\nu \in {}^{\mathbb{F}}\mathbb{N}$. □

We compute the set of naturals in examples 3.0.1.

Corollary 3.19.

(i) *If* $\mathbb{F} = {}^b\mathbb{R}$, *then* ${}^{\mathbb{F}}\mathbb{N} = \mathbb{N}$.

(ii) *If* $\mathbb{F} = M_\rho$ *(or* $\mathbb{F} = P_\omega$*), then* ${}^{\mathbb{F}}\mathbb{N} = \mathbb{N}$.

(iii) *If* $\mathbb{F} = E_\rho$, *then* ${}^{\mathbb{F}}\mathbb{N} = \mathbb{F} \cap {}^*\mathbb{N}$.

More generally, let us define

$$\mathbb{F}^{\#} = \{\alpha \in {}^*\mathbb{R} : \forall R \in \mathbb{F}_+,\ R^{|\alpha|} \in \mathbb{F}_+\}.$$

Clearly, $\mathbb{F}^{\#}$ is a subring of ${}^*\mathbb{R}$. Moreover, we have

$${}^b\mathbb{R} \subset \mathbb{F}^{\#} \subset \mathbb{F} \text{ and } \mathbb{F}^{\#} \cap {}^*\mathbb{N} = {}^{\mathbb{F}}\mathbb{N}.$$

As an example, one can show by direct computations that $M_\rho^{\#} = {}^b\mathbb{R}$.

The following mappings are well defined

(i) $\ln : \mathbb{F}_+ \setminus {}^i\mathbb{F} \to \mathbb{F}$,

(ii) $\ln : {}^*\mathbb{R}_+ \setminus \mathbb{F}_+ \to {}^*\mathbb{R}_+ \setminus \mathbb{F}_+^{\#}$, where $\mathbb{F}_+^{\#} = \mathbb{F}^{\#} \cap {}^*\mathbb{R}_+$,

(iii) $\exp : \mathbb{F}^{\#} \to \mathbb{F}_+$. Moreover, if there exists $\mathbb{G}$ a subring of ${}^*\mathbb{R}$ such that $\mathbb{G} \supset \mathbb{F}^{\#}$ and $\exp(\mathbb{G}) \subset \mathbb{F}_+$, then $\mathbb{F}^{\#} \subset \mathbb{G} \subset \mathbb{F}$.

3.2 Algebraic description of the Robinson field ${}^{\rho}\mathbb{R}$

Given ρ, a positive infinitesimal number, Robinson[14] constructed ${}^{\rho}\mathbb{R}$, the field of real ρ-asymptotic numbers, and defined a valuation topology on ${}^{\rho}\mathbb{R}$.

In this section, we will describe an algebraic construction of such field.

Let R be a valuation ring which is not a field, and Q its fraction field. Thus R is a local ring and, $\mathfrak{m}$ will denote its maximal ideal.

Pick ρ a nonzero element in $\mathfrak{m}$. Let R_ρ be the smallest subring of Q containing R and $\frac{1}{\rho}$.

Proposition 3.20. *R_ρ is the localization of R with respect the multiplicative set $\{1, \rho, \rho^2, \dots, \}$, that is, $R_\rho = \bigcup_{n\geq 1} \frac{1}{\rho^n} R$. Its maximal ideal is given by $\mathfrak{m}_\rho = \bigcap_{n\geq 1} \rho^n R$ and $\rho \notin \mathfrak{m}_\rho$.*

Proof. Clearly $\bigcup_{n\geq 1} \frac{1}{\rho^n} R$ is the smallest subring of Q containing R and ρ. Now, we have only to prove that $\mathfrak{m}_\rho = \bigcap_{n\geq 1} \rho^n R$. Since $\bigcap_{n\geq 1} \rho^n R$ is a proper ideal in R, we have $\bigcap_{n\geq 1} \rho^n R \subset \mathfrak{m}_\rho$. Conversely, let $x \notin \bigcap_{n\geq 1} \rho^n R$. Then there is $n \in \mathbb{N}$ such that $\frac{x}{\rho^n} \notin R$. Since R is a valuation ring, we obtain $\frac{\rho^n}{x} \in R$, i.e., $\frac{1}{x} \in \frac{1}{\rho^n} R \subset R_\rho$. Hence x is invertible in R_ρ and $x \notin \mathfrak{m}_\rho$. Finally $\rho \notin \mathfrak{m}_\rho$, otherwise ρ will be invertible in R. □

The following shows that the latter construction is useful if R is a non-Noetherian ring.

Proposition 3.21. *Suppose that R is Noetherian valuation ring. Then $R_\rho = Q$, the field of fractions of R.*

Proof. This follows from the inclusion $m_\rho \subset \bigcap_{n\geq 1} m^n$ and the Krull's intersection theorem which states that $\bigcap_{n\geq 1} m^n = \{0\}$. □

Let us recall that a Noetherian valuation ring is either a field or a discrete valuation ring, that is, its value group is isomorphic to $\mathbb{Z}$.

Theorem 3.22.

(i) *If $\mathfrak{I}$ is an ideal in R such that $\sqrt{\mathfrak{I}} \subsetneqq \mathfrak{m}$, then $\mathfrak{I} \subset \bigcap_{\rho\in\mathfrak{m}\setminus\sqrt{\mathfrak{I}}} \mathfrak{m}_\rho$. Moreover, if $\mathfrak{I}$ is a radical ideal non maximal, then $\mathfrak{I} = \bigcap_{\rho\in\mathfrak{m}\setminus\mathfrak{I}} \mathfrak{m}_\rho$.*

(ii) *If $\mathfrak{p}$ is a prime ideal in R, non maximal, then $\mathfrak{p} = \bigcap_{\rho\in\mathfrak{m}\setminus\mathfrak{p}} \mathfrak{m}_\rho$.*

(iii) *The maximal ideal $\mathfrak{m} \supset \bigcup_{\rho\in\mathfrak{m}\setminus\{0\}} \mathfrak{m}_\rho$. Furthermore, if $\bigcap_{\rho\in\mathfrak{m}\setminus\{0\}} R_\rho = R$, then $\mathfrak{m} = \bigcup_{\rho\in\mathfrak{m}\setminus\{0\}} \mathfrak{m}_\rho$.*

Proof. (i) Let $z \in \mathfrak{I}$ and assume that there exists $\rho \in \mathfrak{m} \setminus \sqrt{\mathfrak{I}}$ such that $z \notin \mathfrak{m}_\rho$. Then there exists $n \in \mathbb{N}$, $n \geq 1$ such that $\frac{z}{\rho^n} \notin R$, thus $\frac{\rho^n}{z} \in R$. Hence $\rho \in \sqrt{\mathfrak{I}}$, a contradiction. Therefore $\mathfrak{I} \subset \bigcap_{\rho\in\mathfrak{m}\setminus\sqrt{\mathfrak{I}}} \mathfrak{m}_\rho$

Conversely, using the fact that $\rho \notin \mathfrak{m}_\rho$, for any non-zero element of $\mathfrak{m}$, we obtain $\mathfrak{m} \setminus \mathfrak{I} \subset \bigcup_{\rho\in\mathfrak{m}\setminus\mathfrak{I}} \mathfrak{m} \setminus \mathfrak{m}_\rho$. Thus, if $\mathfrak{I}$ is a radical ideal, then $\mathfrak{I} = \bigcap_{\rho\in\mathfrak{m}\setminus\mathfrak{I}} \mathfrak{m}_\rho$.

(ii) Clear.

(iii) Obviously, we have $\bigcup_{\rho\in\mathfrak{m}\setminus\{0\}} \mathfrak{m}_\rho \subset \mathfrak{m}$. Conversely, let $x \in \mathfrak{m}$ and assume that for any ρ, non-zero element of $\mathfrak{m}$, $x \notin \mathfrak{m}_\rho$, then, there exists $n \in \mathbb{N}$ such that $\frac{x}{\rho^n} \notin R$, so its inverse $\frac{\rho^n}{x} \in R$. Hence $\frac{1}{x} \in \bigcap_{\rho\in\mathfrak{m}\setminus\{0\}} R_\rho = R$, a contradiction. □

Corollary 3.23.

(i) *If $\mathfrak{p}$ is a prime ideal in ${}^b\mathbb{R}$, non maximal, then $\mathfrak{p} = \bigcap_{\rho\in\mathfrak{m}\setminus\mathfrak{p}} N_\rho$.*

(ii) *The maximal ideal, ${}^i\mathbb{R} = \bigcup_{\rho\in\mathfrak{m}\setminus\{0\}} N_\rho$.*

For the definition of N_ρ, see Examples 3.0.1 (iii).

Proof. We have only to show that $\bigcap_{\rho>0} M_\rho = {}^b\mathbb{R}$. Let $x \notin {}^b\mathbb{R}$ and

$$\mathcal{A} = \{n \in {}^*\mathbb{N} : |x|^{1/n} > n\}.$$

$\mathcal{A}$ is an internal subset of ${}^*\mathbb{R}$ containing $\mathbb{N}$, hence $\mathcal{A}$ contains some infinite integer $N \in \mathbb{N}^\infty$. Let $\rho := |x|^{-1/N}$. Since $\rho < 1/N$, we get $\rho \in {}^i\mathbb{R}$ and $|x| \geq \rho^{-n}$ for all $n \in \mathbb{N}$, i.e., $x \notin M_\rho$. □

4 Bornologies on ${}^*X$

Definition 4.1. *Let $(X, \mathcal{B})$ be a bornological space. Then $\mathcal{B}_{{}^*X} = \bigcup_{B\in\mathcal{B}} {}^*B$ is called the set of bounded points of ${}^*X$.*

Similarly to the topological context where two topologies are considered on a nonstandard extension of a topological space, we will construct bornologies on a nonstandard extension of a bornological space.

Definition 4.2. *Let $(X, \mathcal{B})$ be a bornological space. Then*

(i) *$({}^*B)_{B\in\mathcal{B}}$ generates a bornology on ${}^*X$ called the S-bornology.*

(ii) *$\{\mathcal{B}_{{}^*X}\}$ generates a bornology on ${}^*X$ called the S-trivial bornology.*

(iii) *${}^*\mathcal{B}$ generates a bornology on ${}^*X$ called the Q-bornology.*

Elements of the S-bornology (resp. Q-bornology) are called S-bounded (resp. Q-bounded).
Hence A a subset of ${}^*X$ is S-bounded if there exists $B \in \mathcal{B}$ such that $A \subset {}^*B$ and A is Q-bounded if there exists $B \in {}^*\mathcal{B}$ such that $A \subset B$.

Trivially, the S-bornology is coarser than the Q-bornology.

Proposition 4.3. *The S-trivial bornology is coarser than the Q-bornology.*

This is a direct consequence of the following lemma, the proof of which requires that $V({}^*X)$ is an enlargement.

Lemma 4.4. *Let $(X, \mathcal{B})$ be a bornological space. Then there exists $B \in {}^*\mathcal{B}$ such that $\mathcal{B}_{{}^*X} \subset B$.*

Proof. The binary relation P on $\mathcal{B}\times\mathcal{B}$ defined by $P(A, B)$ if $A \subset B$ is concurrent. If $A_1, \ldots, A_n \in \mathcal{B}$ then $B = A_1 \cup \ldots \cup A_n$ satisfies $P(A_i, B)$, $1 \leq i \leq n$. Since $V({}^*X)$ is an enlargement, there exists $B \in {}^*\mathcal{B}$, so that ${}^*A \subset B$ for all $A \in \mathcal{B}$ and hence $\mathcal{B}_{{}^*X} \subset B$. □

Lemma 4.4 and the transfer principle imply the following theorem :

Theorem 4.5. *Let $(X, \mathcal{B})$ and $(Y, \mathcal{B}')$ be two bornological spaces and $f : X \to Y$ be a mapping. Then f is bounded if and only if ${}^*f(\mathcal{B}_{{}^*X}) \subset \mathcal{B}'_{{}^*Y}$*

In other words, $f : X \to Y$ is bounded if and only if f sends bounded points of ${}^*X$ to bounded points of ${}^*Y$.

We note that some intermediate bornologies can be considered in the context of metric spaces and locally convex spaces.

For instance, in ${}^*\mathbb{R}$, the collection of subintervals with positive real radii is a base for the S-bornology and ${}^b\mathbb{R}$ is a base the S-trivial bornology. The collection of subintervals with positive hyperreal radii is a base for the Q-bornology. Now, let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{R}$, subintervals with radii in ${}^a\mathbb{F}_+$ is a base for a bornology called the $\mathbb{F}$-bornology.
The bornology generated by $\{\mathbb{F}\}$ is called the $\mathbb{F}$-trivial bornology. Hence

$$S\text{-bornology } \subset \mathbb{F}\text{-bornology } \subset \mathbb{F}\text{-trivial bornology } \subset Q\text{-bornology}.$$

More generally, let (X, d) be a metric space and $\mathbb{F}$ be a convex subring of ${}^*\mathbb{R}$, the collection of balls $\{B_r(x) \subset {}^*X : x \in X, r \in {}^a\mathbb{F}_+\}$ is a base for a bornology on ${}^*X$ called the $\mathbb{F}$-bornology. Besides, when $\mathbb{F} = {}^b\mathbb{R}$, this bornology is reduced to the S-bornology and for $\mathbb{F} = {}^*\mathbb{R}$, we obtain the Q-bornology.

4.1 Bounded Polynomials

Let $n \in \mathbb{N}$, $n \geq 1$. A multi-index $\nu = (\nu_1, \ldots, \nu_n)$ is just an element on $\mathbb{N}^n$ and as usual we define $|\nu| = \nu_1 + \ldots + \nu_n$.
Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\nu = (\nu_1, \ldots, \nu_n)$ a multi-index. If $a_\nu \in \mathbb{C}$ then $a_\nu z^\nu = a_\nu z_1^{\nu_1} \ldots z_n^{\nu_n}$ is called a monomial. A polynomial in n variables is a function $P : \mathbb{C}^n \to \mathbb{C}$ of the form $P(z) = \sum_{|\nu|\leq d} a_\nu z^\nu$, where $d \in \mathbb{N}$ and $a_\nu \in \mathbb{C}$. The set of all polynomials in n variables is denoted by $\mathbb{C}[T_1, \ldots, T_n]$. Every polynomial in ${}^*(\mathbb{C}[T_1, \ldots, T_n])$ is called a nonstandard polynomial. If follows by the transfer principle that every nonstandard polynomial can be written in the form $P(z) = \sum_{|\nu|\leq d} a_\nu z^\nu$, where $a_\nu \in {}^*\mathbb{C}$ and $d \in {}^*\mathbb{N}$. We associate to P the internal polynomial $|P|$ defined by $|P|(z) = \sum_{|\nu|\leq d} |a_\nu| z^\nu$. The set of all internal polynomials in n variables is denoted by $\mathbb{C}[T_1, \ldots, T_n]_{int}$.

We will define several kind of boundedness of internal polynomials and provide characterizations in terms of their coefficients.

Definition 4.6. *Let $P \in \mathbb{C}[T_1, \ldots, T_n]_{int}$ be an internal polynomial and $\mathbb{F}$ and $\mathbb{G}$ be two convex subrings of ${}^*\mathbb{C}$ such $\mathbb{F} \subset \mathbb{G}$. We call P*

(i) *an $(\mathbb{F}, \mathbb{G})$-bounded polynomial if $P(\mathbb{F}^n) \subset \mathbb{G}$,*

(ii) *an absolutely $(\mathbb{F}, \mathbb{G})$-bounded polynomial if $|P|(\mathbb{F}^n) \subset \mathbb{G}$,*

(iii) *an $(\mathbb{F}, \mathbb{G})$-infinitesimal polynomial if $P(\mathbb{F}^n) \subset {}^i\mathbb{G}$.*

In fact, we will prove in Corollary 4.8 that the first two notions of boundedness defined above coincide.

Let ${}^{\mathbb{G}}_{\mathbb{F}}\mathbb{C}[T_1, \ldots, T_n]$ denote the set of $(\mathbb{F}, \mathbb{G})$-bounded internal polynomials. We simply denote by ${}^{\mathbb{F}}\mathbb{C}[T_1, \ldots, T_n] := {}^{\mathbb{F}}_{\mathbb{F}}\mathbb{C}[T_1, \ldots, T_n]$, the set of $(\mathbb{F}, \mathbb{F})$-bounded polynomials and by ${}^{i\mathbb{F}}\mathbb{C}[T_1, \ldots, T_n]$ the set of $(\mathbb{F}, \mathbb{F})$ infinitesimal polynomials.

The following proposition gives a characterization of boundedness of internal entire functions.

Proposition 4.7. *Let $f \in {}^*\mathcal{O}(\mathbb{C}^n)$ be an internal entire function, $f = \sum_{\nu \in {}^*\mathbb{N}^n} a_\nu z^\nu$. Then*

(i) *f is $(\mathbb{F}, \mathbb{G})$-bounded if and only if for any $R \in \mathbb{F}_+$, there exists $C_R \in \mathbb{G}_+$, such that $|a_\nu|\, R^{|\nu|} \leq C_R$, for any $\nu \in {}^*\mathbb{N}^n$.*

(ii) *f is $(\mathbb{F}, \mathbb{G})$-infinitesimal if and only if for any $R \in \mathbb{F}_+$, there exists $C_R \in {}^i\mathbb{G}_+$, such that $|a_\nu|\, R^{|\nu|} \leq C_R$, for any $\nu \in {}^*\mathbb{N}^n$.*

Since the proofs of the two statements are similar, we prove only (i).

Proof. (i) Let $f = \sum_{\nu \in {}^*\mathbb{N}^n} a_\nu z^\nu$ be an $(\mathbb{F}, \mathbb{G})$-bounded entire function and R be any positive number in $\mathbb{F}$. We denote by $T_R = \{(\xi_1, \ldots, \xi_n) \in {}^*\mathbb{C}^n, |\xi_1| = \ldots = |\xi_n| = R\}$. Applying the Cauchy integral formula, we obtain

$$a_\nu = \frac{1}{(2\pi i)^n} \int_{T_R} \frac{f(\xi_1, \ldots, \xi_n)}{\xi_1^{\nu_1+1} \ldots, \xi_n^{\nu_n+1}}\, d\xi_1 \ldots d\xi_n.$$

Again, by transfer, the polynomial f attains its maximum at some point $\xi_R \in T_R$. Since f is $(\mathbb{F}, \mathbb{G})$-bounded, we have $f(\xi_R) \in \mathbb{G}$. Hence there exists $C_R \in \mathbb{G}_+$, such that

$$|a_\nu| \leq \frac{C_R}{R^{|\nu|}}, \quad \forall \nu \in {}^*\mathbb{N}^n.$$

Now, we verify the converse. Let $z = (z_1, \ldots, z_n) \in \mathbb{F}^n$, then there exists $R \in \mathbb{F}_+$, such that $|z_i| \leq R$, for any $i = 1, \ldots, n$. By hypothesis, there exists $C_{2R} \in \mathbb{G}_+$, such that $|a_\nu|(2R)^{|\nu|} \leq C_{2R}$ for each $\nu \in {}^*\mathbb{N}$. Hence

$$|f(z)| \leq \sum_{\nu \in {}^*\mathbb{N}^n} |a_\nu|\, R^{|\nu|} \leq \sum_{\nu \in {}^*\mathbb{N}^n} |a_\nu|(2R)^{|\nu|} \frac{1}{2^{|\nu|}} \leq C_{2R} \sum_{\nu \in {}^*\mathbb{N}^n} \frac{1}{2^{|\nu|}} \leq 2^n\, C_{2R}.$$

Therefore, f is $(\mathbb{F}, \mathbb{G})$-bounded. ☐

Corollary 4.8. *An internal entire function $f \in {}^*\mathcal{O}(\mathbb{C}^n)$ is $(\mathbb{F},\mathbb{G})$-bounded if and only if f is absolutely $(\mathbb{F},\mathbb{G})$-bounded.*

In order to give a characterization of boundedness in terms of the coefficients of the polynomial, we introduce the following

Definition 4.9. *Let $\mathbb{F}$ and $\mathbb{G}$ be two convex subrings of ${}^*\mathbb{C}$ such that $\mathbb{F} \subset \mathbb{G}$. Define*

$$ {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N} := \{\nu \in {}^*\mathbb{N} : \forall R \in \mathbb{F}_+,\ R^\nu \in \mathbb{G}\} $$

the set of $(\mathbb{F},\mathbb{G})$ naturals.

We remark that ${}^{\mathbb{F}}_{\mathbb{F}}\mathbb{N}$, the set of $(\mathbb{F},\mathbb{F})$ naturals, is reduced to ${}^{\mathbb{F}}\mathbb{N}$, the set of naturals in $\mathbb{F}$. Moreover, by convexity of $\mathbb{G}$, we have

$$ \nu \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N} \text{ if and only if } \forall n \in \mathbb{F} \ \cap {}^*\mathbb{N},\ n^\nu \in \mathbb{G}. $$

Example 4.10. *If $\mathbb{F} = {}^b\mathbb{R}$ and $\mathbb{G} = M_\rho$, then*

$$ {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N} = \{\nu \in {}^*\mathbb{N} : \nu \le \alpha|\ln\rho| \text{ for some } \alpha \in \mathbb{R}_+\} = {}^*\mathbb{N} \cap (|\ln\rho|\,{}^b\mathbb{R}_+). $$

Proposition 4.11. *Let $\mathbb{F}$ and $\mathbb{G}$ be two subrings of ${}^*\mathbb{C}$ such that $\mathbb{F} \subset \mathbb{G}$. Then*

(i) ${}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ *is a monoid.*

(ii) $\mathbb{N} \subset {}^{\mathbb{F}}\mathbb{N} \cup {}^{\mathbb{G}}\mathbb{N} \subset {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N} \subset \mathbb{G} \cap {}^*\mathbb{N}$.

(iii) *Let $n, m \in {}^*\mathbb{N}$ such that $m \le n$. If $n \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, then $m \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$.*

(iv) $\left({}^i\mathbb{F}\right)^N \subset {}^i\mathbb{G}$ *for any $N \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$.*

(v) *If ${}^b\mathbb{C} \subsetneq \mathbb{G}$, then $\mathbb{N} \subsetneq {}^{\mathbb{G}}_{{}^b\mathbb{C}}\mathbb{N}$.*

Proof. The properties(i) and (ii) are straightforward.
(iii) Let $\alpha \in {}^i\mathbb{F}$ and $N \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$. Then there is $R_0 \in \mathbb{F}_+$ such that $R_0^N \notin \mathbb{G}$, that is, $\frac{1}{R_0^N} \in {}^i\mathbb{G}$. The number $\alpha R_0 \in {}^i\mathbb{F}$ so it satisfies $|\alpha R_0| < 1$. Hence

$$ |\alpha^N| \le \frac{1}{R_0^N} \in {}^i\mathbb{G}. $$

By convexity of ${}^i\mathbb{G}$, we deduce that $\alpha^N \in {}^i\mathbb{G}$.

(iv) Since ${}^b\mathbb{C} \subsetneq \mathbb{G}$, then there exists $\omega \in \mathbb{G} \cap {}^\infty\mathbb{R}_+$. Let $N := [\ln\omega]$, the greatest integer of $\ln\omega$. Thus $N \in \mathbb{G} \cap {}^\infty\mathbb{N}$ because $N \le \omega$. For any $n \in \mathbb{N}$, we have $n^N \le \omega^n \in \mathbb{G}$. Hence $N \in {}^{\mathbb{G}}_{{}^b\mathbb{C}}\mathbb{N}$. □

Theorem 4.12 (Principles of permanence of ${}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$)**.**
Let A be an internal subset of ${}^*\mathbb{N}$ and $\mathbb{F}$ and $\mathbb{G}$ be two subrings of ${}^*\mathbb{C}$ such that $\mathbb{F} \subset \mathbb{G}$.

(i) *(The Underflow Principle) Let $K \in {}^\infty\mathbb{N}$ be an infinite integer. If every $H \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ with $H \le K$ belongs to A, then there is some $k \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ such that $[\![k..K]\!] \subset A$.*

(ii) *(The Overflow Principle) Let* $k \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$. *If every* $n \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ *with* $n \geq k$ *belongs to* A, *then there is some* $K \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ *such that* $[\![k..K]\!] \subset A$.

Since the proofs are similar to the classical permanence principles, see Goldlblatt [6] p.136, we will give only the proof of the underflow principle.

Proof. (i) Our hypothesis is that $[\![H..K]\!] \subset A$ for all $H \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ with $H \leq K$. Let

$$B = \{k \in {}^*\mathbb{N} \,:\, [\![k..K]\!] \subset A\}.$$

B is a nonempty internal subset of ${}^*\mathbb{N}$. Then by the internal least number principle it has a least element k, and a such k must belong to ${}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ because if $k \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ then $k-1$ would be in ${}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, so by our hypothesis $k-1$ would also be in B but less than k. □

Theorem 4.13. *Let* $P \in \mathbb{C}[T_1, \dots, T_n]_{int}$ *be an internal polynomial, that is,*

$$P = \sum_{|\nu| \leq d} a_\nu T^\nu, \text{ where } a_\nu \in {}^*\mathbb{C}, d \in {}^*\mathbb{N}^\infty.$$

Then P *is* $(\mathbb{F}, \mathbb{G})$*-bounded if and only if the following two conditions are satisfied*

(i) $a_\nu \in \mathbb{G}$ *for* $|\nu| \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$,

(ii) $|a_\nu|^{\frac{1}{|\nu|}} \in {}^i\mathbb{F}$ *for* $|\nu| \notin {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$; $|\nu| \leq d$.

Proof. Let $P = \sum_{|\nu| \leq d} a_\nu T^\nu$ be an $(\mathbb{F}, \mathbb{G})$-bounded polynomial. Then, by Proposition 4.7, there exists $C \in \mathbb{G}_+$ such that $|a_\nu| \leq C$ for each $\nu \in {}^*\mathbb{N}^n$. In particular, $a_\nu \in \mathbb{G}$ for $|\nu| \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$.
Suppose there exists $|\nu_0| \notin {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ such that $|a_{\nu_0}|^{\frac{1}{|\nu_0|}} \notin {}^i\mathbb{F}$. So, there exists $m \in {}^a\mathbb{F}_+$ such that

$$|a_{\nu_0}|^{\frac{1}{|\nu_0|}} \geq m.$$

The condition $|\nu_0| \notin {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ implies that there exists $R_0 \in \mathbb{F}_+$ such that $R_0^{|\nu_0|} \notin \mathbb{G}$. Let $R = \frac{R_0}{m}$. Clearly, $R \in \mathbb{F}$ and for any $C \in \mathbb{G}_+$, we have

$$|a_{\nu_0}| R^{|\nu_0|} \geq R_0^{|\nu_0|} \geq C$$

which contradicts Proposition 4.7(i).

Conversely, let $R \in \mathbb{F}_+$, we set

$$A_R = \{|\nu| \in {}^*\mathbb{N} : |a_\nu|^{\frac{1}{|\nu|}} \leq \frac{1}{R+1}\}.$$

A_R is an internal subset of ${}^*\mathbb{N}$ containing $\{m \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N} : m \leq d\}$. By the permanence principle, there exists $n_0 \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, such that $[\![n_0..d]\!] \subset A_R$. Hence

$$|a_\nu| R^\nu \leq 1 \text{ for every } \nu \in {}^*\mathbb{N}^n, \text{ with } n_0 \leq |\nu| \leq d.$$

For $|\nu| \leq n_0$, we have $a_\nu R^\nu \in \mathbb{G}$, as $R^\nu \in \mathbb{G}$ and $a_\nu \in \mathbb{G}$. Let $M_R = \max_{|\nu| \leq n_0}(|a_\nu| R^\nu)$. $M_R \in \mathbb{G}$ and $|a_\nu| R^{|\nu|} \leq \max(1, M_R)$ for any $\nu \in {}^*\mathbb{N}^n$, $|\nu| \leq d$, which shows that P is $(\mathbb{F}, \mathbb{G})$-bounded. □

Corollary 4.14. *The ring* ${}^{\mathbb{G}}_{\mathbb{F}}\mathbb{C}[T_1, \ldots, T_n]$ *is invariant by any partial derivative* ∂^α *for* $\alpha \in \mathbb{N}^n$.

Let $P \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{C}[T_1, \ldots, T_n]$. The mean value theorem yields, if $x, y \in \mathbb{F}^n$ such that $x - y \in {}^i\mathbb{G}^n$, then $P(x) - P(y) \in {}^i\mathbb{G}$. This allows us to define $\widehat{P}$, the pointwise function associated to P:

$$\widehat{P} : \mathbb{F}^n / {}^i\mathbb{G}^n \to \widehat{\mathbb{G}}.$$

Now we consider the following situation: let $Q = \sum_{|\nu| \leq d} a_\nu T^\nu \in \mathbb{C}[T_1, \ldots, T_n]_{int}$ be an internal polynomial where $a_\nu \in {}^b\mathbb{C}$ and $d \in {}^\infty\mathbb{N}$. A natural question arises : does there exist a proper convex subring $\mathbb{F}$ of ${}^*\mathbb{C}$ such that the polynomial Q is $\mathbb{F}$-bounded?

For $\omega \in {}^\infty\mathbb{N}$, let us denote

$$P_\omega = \{z \in {}^*\mathbb{C} \,:\, \exists R \in \mathbb{R}_+,\ |z| \leq R^\omega\}.$$

P_ω is a convex subring of ${}^*\mathbb{C}$. Moreover, $(P_\omega)_\omega$ is increasing, that is, if $\omega_1, \omega_2 \in {}^\infty\mathbb{N}$ with $\omega_1 < \omega_2$, then $P_{\omega_1} \subset P_{\omega_2}$.

For $d \in {}^\infty\mathbb{N}$, let us define

$$\mathbb{F}_d = \bigcup_{k \geq 0} P_{d^k},$$

where $P_{d^0} = {}^b\mathbb{C}$. Clearly, $\mathbb{F}_d$ is a proper convex subring of ${}^*\mathbb{C}$.

Theorem 4.15. *Let* $Q = \sum_{|\nu| \leq d} a_\nu T^\nu \in \mathbb{C}[T_1, \ldots, T_n]_{int}$ *be an internal polynomial where* $a_\nu \in {}^b\mathbb{C}$ *and* $d \in {}^\infty\mathbb{N}$. *Then for any* $k \in \mathbb{N}$, *we have*

$$Q(P^n_{d^k}) \subset P_{d^{k+1}}.$$

In particular, the polynomial Q *is* $\mathbb{F}_d$*-bounded, i.e.,* $Q(\mathbb{F}^n_d) \subset \mathbb{F}_d$.

Proof. Let $M = \max_{|\nu| \leq d}(|a_\nu|)$. By assumption $M \in {}^b\mathbb{R}$. Let $z = (z_1, \ldots, z_n) \in P^n_{d^k}$, then there exists a real number R, $R > 1$ and $|z_i| \leq R^{d^k}$. Put $\omega = d^k$.

$$|Q(z)| \leq |Q|(R^\omega, \ldots, R^\omega) \leq M \sum_{|\nu| \leq d} (R^\omega)^{|\nu|} \leq M \left(\frac{R^{\omega(d+1)}}{R^\omega - 1} \right)^n \leq 2^n M (R^n)^{\omega d}.$$

This shows that $Q(z) \in P_{d^{k+1}}$. □

4.2 Quasi-standard part of bounded internal entire functions

Let $f \in {}^*\mathcal{O}(\mathbb{C}^n)$ be an $(\mathbb{F}, \mathbb{G})$-bounded internal holomorphic function over ${}^*\mathbb{C}^n$. Then

$$f(z) = \sum_{\nu \in {}^*\mathbb{N}^n} a_\nu z^\nu \quad \text{and} \quad f(\mathbb{F}^n) \subset \mathbb{G}.$$

For $N \in {}^*\mathbb{N}$, let us denote $f_N = \sum_{|\nu|\leq N} a_\nu z^\nu$ the truncation of the series f up to the order N. It is clear that for any $N \in {}^*\mathbb{N}$, the internal polynomial f_N is $(\mathbb{F}, \mathbb{G})$-bounded.

Theorem 4.16. *$f \in {}^*\mathcal{O}(\mathbb{C}^n)$ be an $(\mathbb{F}, \mathbb{G})$-bounded internal holomorphic function. Then for each $N \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, the tail $(f - f_N)$ is $(\mathbb{F}, \mathbb{G})$ infinitesimal, that is,*

$$(f - f_N)(\mathbb{F}^n) \subset {}^i\mathbb{G}.$$

And

$$\widehat{f(z)} = \lim_{N \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}, N \to \infty} \widehat{f_N}(z).$$

$\widehat{f(z)} = \lim_{N \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}, N \to \infty} \widehat{f_N}(z)$ means that for any $\varepsilon \in \widehat{\mathbb{G}}$, $\varepsilon > 0$, there exists $N \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, such that for any $n \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, $n > N$, we have $|\widehat{f(z)} - \widehat{f_n(z)}| < \varepsilon$.

Proof. Let $N \in {}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, then there is $R_0 \in \mathbb{F}_+$, $R_0 \geq 2$ such that $\dfrac{1}{R_0^N} \in {}^i\mathbb{G}$. Let us first show that $\sum_{|\nu|\geq N+1} \frac{1}{R_0^{|\nu|}} \in {}^i\mathbb{G}$. Indeed,

$$\sum_{|\nu|\geq N+1} \frac{1}{R_0^{|\nu|}} \leq \left(\frac{(1/R_0)^{N+1}}{1 - 1/R_0}\right)^n \leq (\frac{1}{R_0^N})^n \in {}^i\mathbb{G}. \quad (\star)$$

Let $z = (z_1, \ldots, z_n) \in \mathbb{F}^n$, then there exists $R \in \mathbb{F}$ such that $|z_i| \leq R$ for any $i = 1, \ldots, n$. By Proposition 4.7, there exists $C \in \mathbb{G}$ such that

$$|a_\nu|(RR_0)^{|\nu|} \leq C \quad \text{for any } \nu \in {}^*\mathbb{N}^n.$$

Hence

$$|(f - f_N)(z)| \leq \sum_{|\nu|\geq N+1} |a_\nu| R^{|\nu|} \leq \sum_{|\nu|\geq N+1} |a_\nu|(RR_0)^{|\nu|} \frac{1}{R_0^{|\nu|}} \leq C \sum_{|\nu|\geq N+1} \frac{1}{R_0^{|\nu|}}.$$

Using the estimate $(\star)$, we find that $(f - f_N)(z) \in {}^i\mathbb{G}$.

We deduce that $\widehat{f(z)} = \lim_{N \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}, N \to \infty} \widehat{f_N}(z)$. Indeed, let $\varepsilon > 0$ in $\widehat{\mathbb{G}}$ and $\varepsilon_1 \in {}^a\mathbb{G}$ such that $\widehat{\varepsilon_1} = \varepsilon$ and $z \in \mathbb{F}$. Consider the set

$$\mathcal{A}_{\epsilon_1, z} = \{k \in {}^*\mathbb{N} : |f(z) - f_k(z)| \leq \varepsilon_1\}.$$

$\mathcal{A}_{\epsilon_1, z}$ is an internal set containing ${}^*\mathbb{N} \setminus {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$. By the permanence principle (see Theorem 4.12), there exists $N \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$ such that for any n greater than N belongs to $A_{\epsilon_1, z}$, that is, for all $n \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, $n \geq N$, we have $|f(z) - f_n(z)| \leq \varepsilon_1$. Hence for all $n \in {}^{\mathbb{G}}_{\mathbb{F}}\mathbb{N}$, $n \geq N$, we have $|\widehat{f(z)} - \widehat{f_n(z)}| \leq \varepsilon$. □

Combining the previous theorem with Proposition 3.18 yields:

Corollary 4.17. *Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{C}$ such that $\widehat{\mathbb{F}}$ has a nontrival real-valued valuation. If $f \in {}^*\mathcal{O}(\mathbb{C}^n)$ is an $\mathbb{F}$-bounded internal holomorphic function over ${}^*\mathbb{C}^n$, $f(z) = \sum_{\nu \in {}^*\mathbb{N}^n} a_\nu z^\nu$. Then $\widehat{f}(z) = \sum_{\nu \in \mathbb{N}^n} \widehat{a_\nu} z^\nu$, for any $z \in \widehat{\mathbb{F}}^n$. Thus, $\widehat{f}$ defines an entire function over $\widehat{\mathbb{F}}^n$.*

Finally, we note that if we impose a more restrictive condition of boundedness on internal holomorphic functions, we get essentially constants.

Theorem 4.18 (Liouville theorem)**.** *Let $f \in {}^*\mathcal{O}(\mathbb{C}^n)$ be an internal holomorphic function over ${}^*\mathbb{C}^n$. Assume that there exists $C \in \mathbb{F}_+$ such that $|f(z)| \leq C$ for any $z \in \mathbb{F}^n$, then $\widehat{f} = \widehat{f(0)}$.*

Proof. Let $f(z) = \sum_{\nu \in {}^*\mathbb{N}^n} a_\nu z^\nu$ be an internal holomorphic function such that $|f(z)| \leq C$ for any $z \in \mathbb{F}^n$. We combine the overflow principle and the Cauchy integral formula to obtain the existence of $R_0 \in {}^\infty\mathbb{F}_+$ such that $|a_\nu| R_0^{|\nu|} \leq C$ for any $\nu \in {}^*\mathbb{N}^n$. Thus for any $R \in \mathbb{F}_+$ and $|\nu| \geq 1$, we have

$$|a_\nu| R^{|\nu|} \leq |a_\nu| R_0^{|\nu|} (R/R_0)^{|\nu|} \leq CR/R_0 \in {}^i\mathbb{F}.$$

It follows from Proposition 4.7 that $f - f(0)$ is $\mathbb{F}$-infinitesimal, thus $\widehat{f} = \widehat{f(0)}$.
□

5 Nonstandard hulls of topological vector spaces

5.1 Nonstandard topologies on ${}^*E$

Let E be a $\mathbb{K}$-topological vector space, where $\mathbb{K}$ stands either for $\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathcal{N}_0$ the filter of neighborhoods of 0 in E. Let ${}^b\mathbb{K}$ be the set of bounded elements of ${}^*\mathbb{K}$ and ${}^i\mathbb{K}$ be the set of infinitesimals of ${}^*\mathbb{K}$. Let $\mathbb{F}$ be a convex subring of ${}^*\mathbb{K}$, that is, ${}^b\mathbb{K} \subset \mathbb{F} \subset {}^*\mathbb{K}$. Let us recall that ${}^a\mathbb{F} = \mathbb{F} \setminus {}^i\mathbb{F}$ denotes the set of appreciable elements of $\mathbb{F}$.

We define a family of topologies on ${}^*E$ parametrized by convex subrings of ${}^*\mathbb{K}$ as follows: for each p in ${}^*E$, let

$$\mathcal{V}_p({}^*E, \mathbb{F}) = \{p + r\ {}^*U \ : \ U \in \mathcal{N}_0 \text{ and } r \in {}^a\mathbb{F}\}.$$

We will often write $\mathcal{V}({}^*E)$ in place of $\mathcal{V}_0({}^*E, \mathbb{F})$.

Proposition 5.1. *$\mathcal{V}({}^*E)$ is a neighborhood basis of zero in the group $({}^*E, +)$.*

Proof. First, we have to show that $\mathcal{V}({}^*E)$ is a filter base on ${}^*E$.

(i) $0 \in r^*U$ for any $U \in \mathcal{N}_0$ and $r \in {}^a\mathbb{F}$.

(ii) For any $U, V \in \mathcal{N}_0$ and $r, s \in {}^a\mathbb{F}$. Let U_0, V_0 be two balanced neighborhoods of 0, such that $U_0 \subset U$ and $V_0 \subset V$. For $W = U_0 \cap V_0$ and $t = \min(|r|, |s|)$, we have $t \in {}^a\mathbb{F}$ and $t^*W \subset r^*U \cap s^*V$.

(iii) For any $U \in \mathcal{N}_0$, there exists $V \in \mathcal{N}_0$ such that $V - V \subset U$. Thus for any $r \in {}^a\mathbb{F}$, we get $r^*V - r^*V \subset r^*U$. □

If (E, τ) is a topological vector space, we denote by $({}^*E, \tau_{\mathbb{F}})$ the topology on ${}^*E$ generated by $\mathcal{V}_p({}^*E, \mathbb{F})$.

We notice that for $\mathbb{F} = {}^b\mathbb{K}$, the topology on ${}^*E$ generated by $\mathcal{V}({}^*E)$ coincides with the topology generated by the zero neighborhood basis $\{{}^*U : U \in \mathcal{N}_0\}$. The latter topology was defined by Henson and Moore in [7].

5.2 $\mathbb{F}-$bounded elements of ${}^*E$

Definition 5.2.

(i) *A point p of ${}^*E$ is $\mathbb{F}$-bounded if, for each neighborhood U of 0, there exists $r \in {}^a\mathbb{F}$ which satisfies $p \in r\,{}^*U$.*

The set of $\mathbb{F}$-bounded elements of ${}^*E$ will be denoted by $\mathbb{F}({}^*E)$.

(ii) *We define the $\mathbb{F}$-halo of 0 by*

$$\mu_{\mathbb{F}}(0) = \bigcap_{U \in \mathcal{N}_0,\, r \in {}^a\mathbb{F}} r\,{}^*U = \bigcap_{r \in {}^a\mathbb{F}} r\,\mu(0),$$

where $\mu(0) = \bigcap_{U \in \mathcal{N}_0} {}^*U$ stands for the classical halo of 0 in ${}^*E$.

(iii) *For any point $p \in {}^*E$, the $\mathbb{F}$-halo of p,*

$$\mu_{\mathbb{F}}(p) = p + \mu_{\mathbb{F}}(0).$$

Remark 5.3.

(i) *The $\mathbb{F}$-halo of p is exactly the closure of p with respect to the topology generated by $\mathcal{V}_p({}^*E, \mathbb{F})$.*

(ii) *$\mu_{\mathbb{F}}(0)$ is closed under addition and under multiplication by elements of $\mathbb{F}$.*

(iii) *The set of $\mathbb{F}$-bounded elements of ${}^*K$ is $\mathbb{F}$, i.e., $\mathbb{F}({}^*\mathbb{K}) = \mathbb{F}$.*

(iv) *The topology generated by $\mathcal{V}_p({}^*\mathbb{K}, \mathbb{F})$ coincides with the QS-topology on ${}^*\mathbb{K}$, see [11].*

Theorem 5.4. *An element p of ${}^*E$ is $\mathbb{F}$-bounded if and only if $\lambda p \in \mu_{\mathbb{F}}(0)$ whenever $\lambda \in {}^i\mathbb{F}$.*

In particular, this shows that

$${}^i\mathbb{F}.\,\mathbb{F}({}^*E) \subset \mu_{\mathbb{F}}(0).$$

Proof. Suppose that p is $\mathbb{F}$-bounded. Let U be a balanced neighborhood of 0. Then $p \in r_0{}^*U$ for some $r_0 \in {}^a\mathbb{F}$. Therefore, $p \in \omega{}^*U$ for every $\omega \in {}^\infty\mathbb{F}$. Given $r \in {}^a\mathbb{F}$ and $\lambda \in {}^i\mathbb{F}$, with $\lambda \neq 0$. Let $\omega_0 = r/\lambda$. Clearly, $\omega_0 \in {}^\infty\mathbb{F}$ and $\lambda p \in \lambda\omega_0{}^*U \subset r{}^*U$. It follows that λp is in $\mu_{\mathbb{F}}(0)$ whenever λ is in ${}^i\mathbb{F}$.
Conversely, if $\lambda p \in \mu_{\mathbb{F}}(0)$ for every λ in ${}^i\mathbb{F}$ and if U is a neighborhood of 0, then the internal set $\mathcal{A} = \{\omega \in {}^*\mathbb{R} : p \in \omega{}^*U\}$ contains ${}^\infty\mathbb{F}$. Thus by the underflow principle, $\mathcal{A}$ must contain $r \in {}^a\mathbb{F}$. Therefore, the condition implies that p is $\mathbb{F}$-bounded. □

The following is an immediate consequence of Theorem 5.4 and Remark 5.3 (ii).

Corollary 5.5.

(i) *If* $p \in \mathbb{F}({}^*E)$*, then* $\mu_{\mathbb{F}}(p) \subset \mathbb{F}({}^*E)$*.*

(ii) $\mathbb{F}({}^*E)$ *is an* $\mathbb{F}$*-module.*

Theorem 5.6.

(i) $\mathbb{F}({}^*E)$ *is a topological* $\mathbb{F}$*-module, that is, the addition* $\mathbb{F}({}^*E) \times \mathbb{F}({}^*E) \to \mathbb{F}({}^*E)$ *and the scalar multiplication* $\mathbb{F} \times \mathbb{F}({}^*E) \to \mathbb{F}({}^*E)$ *are continuous.*

(ii) $\mathbb{F}({}^*E)$ *is closed in* ${}^*E$*.*

Proof. (i) We have to check that the scalar multiplication $(\lambda, x) \mapsto \lambda x$ satisfies the following conditions, see Warner[22] page 86 :
(TM1) $(\lambda, x) \mapsto \lambda x$ is continuous at $(0, 0)$,
(TM2) for each $c \in \mathbb{F}({}^*E)$, $\lambda \mapsto \lambda c$ is continuous at 0,
(TM3) for each $\alpha \in \mathbb{F}$, $x \mapsto \alpha x$ is continuous at 0.

Given $U \in \mathcal{N}_0$, there exists U_0 a balanced neighborhood of 0 such that $U_0 \subset U$. Let $r \in {}^a\mathbb{F}_+$.
(TM1) : $(|\lambda| \leq 1)(r\,{}^*U_0) \subset r\,{}^*U_0 \subset r\,{}^*U$.
(TM2): Let $c \in \mathbb{F}({}^*E)$, then there exists $r_0 \in {}^a\mathbb{F}$, such that $c \in r_0{}^*U_0$. We have $r/r_0 \in {}^a\mathbb{F}_+$ and $(|\lambda| \leq |r|/|r_0|)\, c \subset r\,(|\lambda| \leq 1){}^*U_0 \subset r\,{}^*U$.
(TM3) Let $\alpha \in \mathbb{F}$, we have $\frac{r}{|\alpha|+1} \in {}^a\mathbb{F}_+$ and $\alpha\, \frac{r}{|\alpha|+1}\, {}^*U_0 \subset r\,{}^*U$.

(ii) To see that $\mathbb{F}({}^*E)$ is closed, let $x \in {}^*E$ with $x \notin \mathbb{F}({}^*E)$. Then there exists U a neighborhood of 0 in E such that $x \notin r\,{}^*U$ for any $r \in {}^a\mathbb{F}_+$. Let O be a balanced neighborhood of 0 such that $O - O \subset U$. Let $V := x + {}^*O$. It follows that V is a neighborhood of x in ${}^*E$ satisfying $V \cap \mathbb{F}({}^*E) = \emptyset$.

Indeed, assume that there exists $y \in V \cap \mathbb{F}({}^*E)$. Then we find $r_0 \in {}^a\mathbb{F}$ with $y \in r_0\,{}^*O$. Since ${}^*O$ is balanced, this implies $x \in r_0{}^*O - {}^*O \subset (|r_0| + 1)({}^*O - {}^*O) \subset (|r_0| + 1){}^*U$, a contradiction. □

For each $U \in \mathcal{N}_0$ and $r \in {}^a\mathbb{F}$ define

$$V_{r,U} = \{(x, y) \in {}^*E \times {}^*E \;:\; x - y \in r\,{}^*U\}$$

Let $\mathcal{U}_{\mathbb{F}}$ be the filter on ${}^*E \times {}^*E$ generated by the filter base $\{V_{r,U} : U \in \mathcal{N}_0,\, r \in {}^a\mathbb{F}\}$. Then $\mathcal{U}_{\mathbb{F}}$ is a translation-invariant uniformity on ${}^*E$ which determine the topology on ${}^*E$ generated by $\mathcal{V}_p({}^*E, \mathbb{F})$.

Theorem 5.7. *If* $\mathbb{F}$ *is generated by an asymptotic scale, then* $({}^*E, \mathcal{U}_{\mathbb{F}})$ *is complete.*

Proof. Assume that $\mathbb{F}$ is generated by the asymptotic scale λ_n. Let $\mathcal{G}$ be a Cauchy filter on ${}^*E$. Then for each $n \in \mathbb{N}$ and each $U \in \mathcal{N}_0$, there exists $F_{n,U} \in \mathcal{G}$ such that

$$F_{n,U} - F_{n,U} \subset \lambda_n{}^*U.$$

Choose some $x_{n,U} \in F_{n,U}$ and consider the system of internal sets $\mathcal{A}_{n,U} := x_{n,U} + \lambda_n{}^*U$. As $F_{n,U} \subset \mathcal{A}_{n,U}$, then $\mathcal{A}_{n,U}$ has the finite intersection property. Hence, by the saturation property, we conclude that $\cap \mathcal{A}_{n,U}$ contains some element $x \in {}^*E$.

We claim that the filter $\mathcal{G}$ converges to x, that is, any neighborhood of x belongs to $\mathcal{G}$.

Let W be any neighborhood of x, then there exists $n \in \mathbb{N}$ and $U \in \mathcal{N}_0$ such that $x + \lambda_n{}^*U \subset W$. Let V be a neighborhood of 0 such that $V - V \subset U$. We have

$$F_{V,n} \subset x_{n,V} + \lambda_n{}^*V \subset (x_{n,V} - x) + (x + \lambda_n{}^*V) \subset -\lambda_n{}^*V + (x + \lambda_n{}^*V) \subset x + \lambda_n{}^*U$$

Hence $x + \lambda_n{}^*U \in \mathcal{G}$ and so $W \in \mathcal{G}$, as claimed. □

Using Theorem 5.6 (ii), we deduce the following

Corollary 5.8. *If $\mathbb{F}$ is generated by an asymptotic scale, then $\mathbb{F}({}^*E)$ is complete.*

Theorem 5.9. *Let (G, τ) and (H, τ') be $\mathbb{K}$-topological vector spaces and $f : G \to H$ be a linear mapping. Consider the following*

(i) *f is continuous at 0.*

(ii) ${}^*f(\mu_{\mathbb{F}}^{\tau}(0)) \subset \mu_{\mathbb{F}}^{\tau'}(0)$.

(iii) ${}^*f\,(\mathbb{F}({}^*G)) \subset \mathbb{F}({}^*H)$.

Then (i) $\iff$ (ii) $\implies$ (iii).
Furthermore, if $\mathbb{F}$ is generated by an asymptotic scale, then (i) $\iff$ (ii) $\iff$ (iii).

Before giving the proof, we need the following lemmas

Lemma 5.10. *Let E be $\mathbb{K}$-topological vector space. Then there exists W, a $*$-open neighborhood of 0, such that $W \subset \mu_{\mathbb{F}}(0)$.*

Proof. According to the saturation principle, there exists V, a $*$- open neighborhood of 0, such that $V \subset \mu(0)$, see [8]. Let α be a nonzero element in ${}^i\mathbb{F}$. We claim that $W := \alpha V$ is a $*$-open neighborhood of 0 satisfying $W \subset \mu_{\mathbb{F}}(0)$. Indeed, the transfer principle shows that W is a *-open neighborhood of 0. Furthermore, for any $r \in {}^a\mathbb{F}$ and for any U, a balanced neighborhood 0, we have $\alpha V \subset \alpha\,{}^*U \subset r\,{}^*U$, thus, $W \subset \mu_{\mathbb{F}}(0)$. □

We prove the converse of Theorem 5.4 under some additional assumptions on $\mathbb{F}$.

Lemma 5.11. *Assume that $\mathbb{F}$ is generated by an asymptotic scale. Then, for each $p \in \mu_{\mathbb{F}}(0)$ there exists $\omega \in {}^{\infty}\mathbb{F}$ such that $\omega p \in \mu_{\mathbb{F}}(0)$. Hence*

$$ {}^{i}\mathbb{F} \,.\, \mu_{\mathbb{F}}(0) = \mu_{\mathbb{F}}(0). $$

Proof. Let $p \in \mu_{\mathbb{F}}(0)$. For each $n \in \mathbb{N}$ and $U \in \mathcal{N}_0$ define the internal set $A(n, U)$ by

$$ A(n,U) = \{x \in {}^*\mathbb{R} \,:\, x \geq \lambda_{-n} \text{ and } xp \in \lambda_n {}^*U\}. $$

Since $\mu_{\mathbb{F}}(0)$ is closed by multiplication by elements of $\mathbb{F}$, each set $A(n, U)$ is nonempty. It follows that the sets $A(n, U)$ is a collection of internal subsets of ${}^*\mathbb{R}$ which has the finite intersection property. Hence, by the saturation principle, there is ω in the intersection of the collection. That is, $\omega \in {}^{\infty}\mathbb{F}$ and satisfies $\omega p \in \lambda_n {}^*U$ for each $n \in \mathbb{N}$ and each $U \in \mathcal{N}_0$. It follows that $\omega p \in \mu_{\mathbb{F}}(0)$, which completes the proof. □

Remark 5.12. *We remark that if $(E, |.|)$ is a normed space then ${}^{i}\mathbb{F} \,.\, \mu_{\mathbb{F}}(0) = \mu_{\mathbb{F}}(0)$ holds for any $\mathbb{F}$ a convex subring of ${}^*\mathbb{R}$. Indeed, let p be a nonzero element in $\mu_{\mathbb{F}}(0)$, i.e., $|p| \in {}^{i}\mathbb{F}$. Let $\omega = 1/\sqrt{|p|}$. Clearly, $\omega \in {}^{\infty}\mathbb{F}$ and $\omega p \in \mu_{\mathbb{F}}(0)$.*

Proof. (Theorem 5.9)

(i) $\implies$ (ii) f is continuous at 0. So for any V neighborhood of 0 in H there exists U, a neighborhood of 0 in G, such that $f(U) \subset V$. By the transfer principle, we get ${}^*f(r\,{}^*U) \subset r\,{}^*V$, for any $r \in {}^{a}\mathbb{F}$. Hence ${}^*f(\mu^{\tau}_{\mathbb{F}}(0)) \subset \mu^{\tau'}_{\mathbb{F}}(0)$.

(ii) $\implies$ (i) Conversely, assume that ${}^*f(\mu^{\tau}_{\mathbb{F}}(0)) \subset \mu^{\tau'}_{\mathbb{F}}(0)$. Let V be an arbitrary neighborhood of 0 in H. Using Lemma 5.10, we obtain :

$$ \text{There exists } W, \text{a *-open nieghborhood of } 0, \text{such that } {}^*f(W) \subset {}^*V. $$

The transfer principle shows that f is continuous.

(ii) $\implies$ (iii) Let $p \in \mathbb{F}({}^*G)$. According to Theorem 5.4, the condition ${}^*f(p) \in \mathbb{F}({}^*H)$ is equivalent to ${}^{i}\mathbb{F}.{}^*f(p) \in \mu^{\tau'}_{\mathbb{F}}(0)$. Indeed, let $\lambda \in {}^{i}\mathbb{F}$,

$$ \lambda^* f(p) \; = {}^*f(\lambda p) \in {}^*f(\mu^{\tau}_{\mathbb{F}}(0)) \subset \mu^{\tau'}_{\mathbb{F}}(0), $$

which completes the proof.

(iii) $\implies$ (ii) Assume that $\mathbb{F}$ is generated by an asymptotic scale. By Lemma 5.10, we have

$$ {}^*f(\mu^{\tau}_{\mathbb{F}}(0)) = {}^*f({}^{i}\mathbb{F}.\,\mathbb{F}({}^*G)) = {}^{i}\mathbb{F}.\,{}^*f(\mathbb{F}({}^*G)) \subset {}^{i}\mathbb{F}.\,\mathbb{F}({}^*H) \subset \mu^{\tau'}_{\mathbb{F}}(0). $$

□

Corollary 5.13. *If τ and τ' are two vectors topologies on E and $\mathbb{F}$ is convex subring of ${}^*\mathbb{R}$ generated by an asymptotic scale. Then*

(i) $\tau \subset \tau' \iff \mathbb{F}_{\tau}({}^*E) \supset \mathbb{F}_{\tau'}({}^*E)$

(ii) $\tau = \tau' \iff \mathbb{F}_\tau({}^*E) = \mathbb{F}_{\tau'}({}^*E)$

Definition 5.14. *Let (E, τ) be a $\mathbb{K}$-topological vector space. The $\mathbb{F}$-nonstandard hull of E is the vector space $\widehat{E}$ defined by*

$$\widehat{E} = \widehat{E}_{\mathbb{F}} = \mathbb{F}({}^*E)/\mu_{\mathbb{F}}(0)$$

equipped with $\widehat{\tau_{\mathbb{F}}}$, the quotient topology of $\tau_{\mathbb{F}}$ on $\mathbb{F}({}^*E)$.

The canonical mapping of $\mathbb{F}({}^*E)$ on $\widehat{E}$ will be denoted by π, thus $\pi(p) = p + \mu_{\mathbb{F}}(0)$ for all $p \in \mathbb{F}({}^*E)$.

We remark that the quotient topology on $\widehat{\mathbb{F}}$ coincides with the (product of) order topology, that is,

$$\widehat{B}(0, r) = \{\alpha \in \widehat{\mathbb{F}} \ : \ |\alpha| < r\}, \quad r \in \widehat{\mathbb{F}}_+,$$

is a neighborhood basis of 0 is $\widehat{\mathbb{F}}$.

Proposition 5.15. *The quotient mapping $\pi : \mathbb{F}({}^*E) \to \widehat{E}$ is continuous and open.*

By Theorem 5.6 and the universal property of the quotient topology, we have

Theorem 5.16. *$\widehat{E}$ is a Hausdorff topological $\widehat{\mathbb{F}}$-vector space.*

$$\begin{array}{ccc} \mathbb{F}({}^*E) \times \mathbb{F}({}^*E) & \xrightarrow{+} & \mathbb{F}({}^*E) \\ \Big\downarrow{\scriptstyle \pi\times\pi} & & \Big\downarrow{\scriptstyle \pi} \\ \widehat{E} \times \widehat{E} & \xrightarrow{\quad + \quad} & \widehat{E} \end{array} \qquad \begin{array}{ccc} \mathbb{F} \times \mathbb{F}({}^*E) & \xrightarrow{\cdot} & \mathbb{F}({}^*E) \\ \Big\downarrow{\scriptstyle \widehat{\mathrm{st}}\times\pi} & & \Big\downarrow{\scriptstyle \pi} \\ \widehat{\mathbb{F}} \times \widehat{E} & \xrightarrow{\quad \cdot \quad} & \widehat{E} \end{array}$$

Theorem 5.17. *If $\mathbb{F}$ is generated by an asymptotic scale, then $\widehat{E}$ is complete.*

Proof. Let $\widehat{\mathcal{G}}$ be a Cauchy filter on $\widehat{E}$ and let $\mathcal{G}$ be the filter on $\mathbb{F}({}^*E)$ generated by $\pi^{-1}(\widehat{\mathcal{G}})$. One can easily check that $\pi^{-1}(\widehat{\mathcal{G}})$ is a Cauchy filter on $\mathbb{F}({}^*E)$, hence by Corollary 5.8, it converges to some $x \in \mathbb{F}({}^*E)$. The continuity of the mapping π implies that $\widehat{\mathcal{G}} = \pi(\mathcal{G})$ converges to $\pi(x)$. □

Proposition 5.18. *If E is a normed space, then topology of $\widehat{\tau_{\mathbb{F}}}$ induces on E the discrete topology.*

Proof. If E is Hausdorff, then E is a subspace of $\widehat{E}$. Indeed, let $x \in E$ such that $\pi(x) = 0$. Hence $x \in \mu_{\mathbb{F}}(0) \cap E \subset \mu(0) \cap E = \overline{\{0\}} = \{0\}$. □

Using Theorem 5.9, we deduce the following

Theorem 5.19. *Let G and H be two $\mathbb{K}$-topological vector spaces and $f : G \to H$ be a continuous linear mapping. Then f gives rise to a continuous $\widehat{\mathbb{F}}$-linear mapping $\widehat{f} : \widehat{G} \to \widehat{H}$ defined by*

$$\widehat{f}(\widehat{x}) = \widehat{f(x)} \quad \textit{for all } \ \widehat{x} \in \widehat{G}.$$

6 Nonstandard hulls of locally convex spaces

Let E be a locally convex topological vector space topologized through a family of seminorms $(p_j)_{j\in J}$. Then $\mathbb{F}({}^*E)$, the set of $\mathbb{F}$-bounded points of ${}^*E$ defined in section 5.2, is given by

$$\mathbb{F}({}^*E) = \{x \in {}^*E : p_j(x) \in \mathbb{F} \text{ for all } j \in J\},$$

and

$$\mu_{\mathbb{F}}(0) = \{x \in {}^*E : p_j(x) \in {}^i\mathbb{F} \text{ for all } j \in J\}.$$

The topology $({}^*E, \tau_{\mathbb{F}})$ is generated by $\{p_j^{-1}(0, r) : r \in {}^a\mathbb{F}_+\}$ as a subbase. In other words, $\tau_{\mathbb{F}}$ coincides with the initial topology on ${}^*E$ making all ${}^*p_j$: ${}^*E \longrightarrow {}^*\mathbb{R}_+$ continuous, where ${}^*\mathbb{R}$ is equipped with the QS-topology generated by $\mathbb{F}$.

The family of seminorms p_j induces on $\widehat{E}$

$$\widehat{p_j} : \widehat{E} \to \widehat{\mathbb{F}}_+.$$

The quotient topology $\widehat{\tau_{\mathbb{F}}}$ on $\widehat{E}$ coincides with the initial topology making all the seminorms $\widehat{p_j} : \widehat{E} \to \widehat{\mathbb{F}}_+$ continuous, where $\widehat{\mathbb{F}}$ is equipped with the order topology.

6.1 Examples

(i) Let $E = \mathcal{E}(\Omega)$, the space of smooth functions over Ω, an open subset of $\mathbb{R}^n$. E is topologized through the family of seminorms $p_{K_i,j}(f) = \sup_{x\in K_i, |\alpha|\leq j} |\partial^\alpha f(x)|$, where $(K_i)_{i\in\mathbb{N}}$ is an exhausting sequence of compact subsets of Ω.

$$\mathbb{F}({}^*\mathcal{E}(\Omega)) = \{f \in {}^*\mathcal{E}(\Omega) : \partial^\alpha f(\text{ns}({}^*\Omega)) \subset \mathbb{F} \text{ for all } \alpha \in \mathbb{N}^n\},$$

$$\mu_{\mathbb{F}}(0) = \{f \in {}^*\mathcal{E}(\Omega) : \partial^\alpha f(\text{ns}({}^*\Omega)) \subset {}^i\mathbb{F} \text{ for all } \alpha \in \mathbb{N}^n\},$$

where $\text{ns}({}^*\Omega)$ stands for the nearstandrad points of ${}^*\Omega$.

The space $\widehat{\mathcal{E}(\Omega)} = \mathbb{F}({}^*\mathcal{E}(\Omega))/\mu_{\mathbb{F}}(0)$ was studied in details in [20] as the nonstandard counterpart of Colombeau algebras.

(ii) Let $(X, \mathcal{O}_X)$ be a separable analytic space, and $U \subset X$ any open set. Recall that $\mathcal{O}_X(U)$ has a structure of a Fréchet space defined by the topology of compact convergence.

$$\mathbb{F}({}^*\mathcal{O}_X(U)) = \{f \in {}^*\mathcal{O}_X(U) : f(\text{ns}({}^*U)) \subset \mathbb{F}\},$$

$$\mu_{\mathbb{F}}(0) = \{f \in {}^*\mathcal{O}_X(U) : f(\text{ns}({}^*U)) \subset {}^i\mathbb{F}\}.$$

The space $\widehat{\mathcal{O}_X(U)} = \mathbb{F}({}^*\mathcal{O}(U))/\mu_{\mathbb{F}}(0)$ is the $\mathbb{F}$-nonstandard hull of $\mathcal{O}_X(U)$. Moreover, the mapping $U \mapsto \widehat{\mathcal{O}_X(U)}$ defines a separated presheaf on X as any element $f \in \widehat{\mathcal{O}_X(U)}$ gives a pointwise mapping $\widehat{f} : \text{ns}({}^*U)/{}^i\mathbb{F}^n \to \widehat{\mathbb{F}}$.

Remark. *Let $\mathcal{G}$ be a Fréchet sheaf on a topological space X with a countable topology, that is, $\mathcal{G}$ is a sheaf of vector spaces on X such that $\mathcal{G}(U)$ is a Fréchet space for every $U \subset X$, an open subset of X, and for every $V \subset U$, the restriction-homomorphism $\mathcal{G}(U) \to \mathcal{G}(V)$ is continuous. Let $\widehat{\mathcal{G}(U)}$ be the $\mathbb{F}$-nonstandard hull of $\mathcal{G}(U)$. If $\widehat{\mathbb{F}}$ has a nontrivial real-valued valuation compatible with its order, then the space $\widehat{\mathcal{G}(U)}$ is a complete with respect to a countable sequence of ultra-seminorms. Hence, using Theorem 5.19, we deduce that the mapping $U \mapsto \widehat{\mathcal{G}(U)}$ gives a Fréchet presheaf on X.*

A fundamental example of a Fréchet sheaf is given by a coherent $\mathcal{O}_X$-module, where $(X, \mathcal{O}_X)$ is a complex space.

(iii) Let us consider $\mathbb{Z}$, the ring of the integers equipped with $|.|_p$, the $p-$adic norm. Recall that if $n \in \mathbb{Z} \setminus \{0\}$, $|n|_p = p^{-\nu_p(n)}$, where ν_p denotes the $p-$adic valuation for $\mathbb{Z}$. Let $\mathbb{F}$ be a proper convex subring of ${}^*\mathbb{R}$, that is, ${}^b\mathbb{R} \subset \mathbb{F} \subsetneq {}^*\mathbb{R}$.
Then the set of $\mathbb{F}$-bounded and $\mathbb{F}$-infinitesimal elements are giving by

$$\mathbb{F}({}^*\mathbb{Z}) = \{n \in {}^*\mathbb{Z} \,:\, |n|_p \in \mathbb{F}\} = {}^*\mathbb{Z},$$

$$\mu_{\mathbb{F}}(0) = \{n \in {}^*\mathbb{Z} \,:\, |n|_p \in {}^i\mathbb{F}\}.$$

One can easily check that $\mu_{\mathbb{F}}(0)$ is an external prime ideal in ${}^*\mathbb{Z}$.

The following proposition provides a generalization of Theorem 18.4.1 in Goldblatt [6], where the author considered the case $\mathbb{F} = {}^b\mathbb{R}$.

Proposition. *Let n be a nonzero hyperinteger $n \in {}^*\mathbb{Z}$. The following are equivalent*

(i) $|n|_p \in {}^i\mathbb{F}$.
(ii) $\nu_p(n) \in {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.
(iii) *n is divisible by p^k for all $k \in {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.*
(iv) *n is divisible by p^K for some $K \in {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.*

The proof is a direct consequence of the following elementary lemma

Lemma. $\log({}^\infty\mathbb{F}_+) \cap {}^*\mathbb{N} = {}^*\mathbb{N} \setminus {}^{\mathbb{F}}_{{}^b\mathbb{R}}\mathbb{N}$.

Proof. (Proposition)
(i) $\Longleftrightarrow$ (ii) : $|n|_p \in {}^i\mathbb{F} \Longleftrightarrow p^{\nu_p(n)} \in {}^\infty\mathbb{F}_+ \Longleftrightarrow \nu_p(n) \in \log({}^\infty\mathbb{F}_+) \cap {}^*\mathbb{N}$.
(ii) $\Longleftrightarrow$ (iii): $p^k/n \Longleftrightarrow k \leq \nu_p(n)$.
(iii) $\Longleftrightarrow$ (vi) : Follows from the overflow principle, see Theorem 4.12. □

Let us consider θ_p the following homomorphism of rings

$$\theta_p : {}^*\mathbb{Z} \longrightarrow \varprojlim_{k \in {}^{\mathbb{F}}_{b_{\mathbb{R}}}\mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z}$$

defined by $\theta_p(n) = (n \mod p^k)_{k \in {}^{\mathbb{F}}_{b_{\mathbb{R}}}\mathbb{N}}$. Using Proposition 6.1, we deduce that $\ker \theta_p = \mu_{\mathbb{F}}(0)$. Hence

$$\widehat{\mathbb{Z}}^{\mathbb{F}} \hookrightarrow \varprojlim_{k \in {}^{\mathbb{F}}_{b_{\mathbb{R}}}\mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z}.$$

where $\widehat{\mathbb{Z}}^{\mathbb{F}}$ denotes the $\mathbb{F}$-nosntandard hull de ${}^*\mathbb{Z}$, that is, $\widehat{\mathbb{Z}}^{\mathbb{F}} = \mathbb{F}({}^*\mathbb{Z})/\mu_{\mathbb{F}}(0) = {}^*\mathbb{Z}/\mu_{\mathbb{F}}(0)$.

We obtain the following commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{Z}}^{\mathbb{F}} & \hookrightarrow & \varprojlim_{k \in {}^{\mathbb{F}}_{b_{\mathbb{R}}}\mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z} \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \widehat{\mathbb{Z}} & \xrightarrow{\simeq} & \varprojlim_{k \in \mathbb{N}} {}^*\mathbb{Z}/p^k\,{}^*\mathbb{Z} \end{array}$$

Here, $\widehat{\mathbb{Z}}$ denotes the ${}^b\mathbb{R}$-nonstandard hull of ${}^*\mathbb{Z}$, which is isomorphic to $\mathbb{Z}_p$, the ring of $p-$adic integers, see Goldblatt [6].

A Spilling Principles

We recall several spilling principles in terms of a *proper* convex subring $\mathbb{F}$ of ${}^*\mathbb{R}$. We should note that the familiar underflow and overflow principles in nonstandard analysis follow as a particular case for $\mathbb{F} = {}^b\mathbb{R}$.

Theorem A.1 (Spilling Principles)**.** *[20] Let $\mathbb{F}$ be a proper convex subring of ${}^*\mathbb{R}$ and $\mathcal{A} \subset {}^*\mathbb{R}$ be an internal set. Then:*

(i) *Overflow of $\mathbb{F}$: If $\mathcal{A}$ contains arbitrarily large numbers in $\mathbb{F}$, then $\mathcal{A}$ contains arbitrarily small numbers in ${}^*\mathbb{R} \setminus \mathbb{F}$. In particular,*

$$\mathbb{F} \setminus {}^i\mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap ({}^*\mathbb{R} \setminus \mathbb{F}) \neq \emptyset$$

(ii) *Underflow of $\mathbb{F} \setminus {}^i\mathbb{F}$: If $\mathcal{A}$ contains arbitrarily small numbers in $\mathbb{F} \setminus {}^i\mathbb{F}$, then $\mathcal{A}$ contains arbitrarily large numbers in ${}^i\mathbb{F}$. In particular,*

$$\mathbb{F} \setminus {}^i\mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap {}^i\mathbb{F} \neq \emptyset$$

(iii) *Overflow of* ${}^{i}\mathbb{F}$ *: If* $\mathcal{A}$ *contains arbitrarily large numbers in* ${}^{i}\mathbb{F}$*, then* $\mathcal{A}$ *contains arbitrarily small numbers in* $\mathbb{F} \setminus {}^{i}F$*. In particular,*

$${}^{i}\mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathbb{F} \setminus {}^{i}\mathbb{F}) \neq \emptyset$$

(iv) *Underflow of* ${}^{*}\mathbb{R} \setminus \mathbb{F}$ *: If* $\mathcal{A}$ *contains arbitrarily small numbers in* ${}^{*}\mathbb{R} \setminus \mathbb{F}$*, then* A *contains arbitrarily large numbers in* $\mathbb{F}$*. In particular,*

$${}^{*}\mathbb{R} \setminus \mathbb{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathbb{F} \setminus {}^{i}\mathbb{F}) \neq \emptyset$$

We should mention that these spilling principles *fail* if $\mathbb{F} = {}^{*}\mathbb{R}$ and ${}^{i}\mathbb{F} = \{0\}$.

Acknowledgments

We would like to acknowledge the support provided by the Deanship of Scientific Research (DSR) at the King Fahd University of Petroleum and Minerals (KFUPM) for funding this work through project No IN121034.

References

[1] N. Bourbaki, *General Topology. Chapters 5–10*, Elements of Mathematics (Berlin), Springer-Verlag, 1989

[2] N. Bourbaki, *Commutative Algebra*, Addison-Wesley, 1972.

[3] A. Delcroix and D. Scarpalezos, *Topology on Asymptotic Algebras of Generalized Functions and Applications*, Mh. Math. **129**, (2000) 1-14

[4] R. Estrada and R. P. Kanwal, *Asymptotic Analysis: A Distributional Approach*, Birkhauser, Boston, Basel, Berlin, 1994.

[5] D. S. Jones, *Introduction to asymptotics. A treatment using nonstandard analysis.* World Scientific Publishing Co., Inc., River Edge, NJ, 1997

[6] R. Goldblatt, *Lectures on the Hyperreals*, Springer-Verlag New York, 1998.

[7] Henson, C. Ward and Moore, Jr., L. C., *The nonstandard theory of topological vector spaces*, Trans. Amer. Math. Soc.**172**, (1972) 405-435

[8] A.E. Hurd and P.A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, 1985

[9] Hogbe-Nlend H., *Théorie des bornologies et applications*, (French) Lecture Notes in Mathematics, Vol. 213, Springer-Verlag, 1971.

[10] Hogbe-Nlend H., *Bornologies and functional analysis. Introductory course on the theory of duality topology-bornology and its use in functional analysis*, North-Holland Mathematics Studies, Vol. 26, 1977.

[11] A.Khalfallah, *New nonstandard topologies*, Monatsh. Math. **172**, 3-4, (2013) 323–344

[12] A. Khalfallah and S. Kosarew, *Complex spaces and nonstandard schemes*, J. Log. Anal.,**2**, (2010), paper 9, 1-60.

[13] T. Lindstrom, *An invitation to nonstandard analysis, in: Nonstandard Analysis and its Applications*, N. Cutland (Ed), Cambridge U. Press, 1988, p. 1-105

[14] A. H. Lightstone and A. Robinson, *Nonarchimedean Fields and Asymptotic Expansions*, North-Holland, Amsterdam, 1975.

[15] W. A. J. Luxemburg, *A general theory of monads, in Applications of Model Theory to Algebra*, Analysis and Probability, Rinehart and Winston, New York, 1969, pp. 18-86

[16] W. A. J. Luxemburg, *On a class of valuation fields introduced by Robinson*, Israel J. Math. **25** (1976), p. 189-201.

[17] P. Ribenboim, *The Theory of Classical Valuation*, Springer Monographs in Mathematics, Springer-Verlag, 1999

[18] A. Robinson, *Nonstandard Analysis*, North Holland, Amsterdam, 1966.

[19] K. D. Stroyan and W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York, 1976.

[20] T. D. Todorov, *Lecture Notes: Non-Standard Approach to J.F. Colombeau's Theory of Generalized Functions*, arXiv:1010.3482

[21] I. Van den Berg, *Nonstandard asymptotic analysis.* Lecture Notes in Mathematics, **1249**. Springer-Verlag, Berlin, 1987

[22] S. Warner, *Topological rings*, North-Holland Mathematics Studies Vol. **178**, 1993